Origami, Algebra, and the Cubic

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Introduction. The Japanese art of paper folding has a long history, though its foundations in geometry and algebra have only been explored over the past hundred years or so. In order to begin to reveal some of the mathematics underlying origami we will first examine what folds are allowed in origami, what lengths can be constructed, what points can be located, what its relationship is to classical straightedge and compass constructions, and what origami can do that cannot be done with only a straightedge and compass.

Constructibility. To begin with, we will need a few basic definitions — we will need to know what a starting state of a sheet of paper is, and what it is we mean for a point or a line to be constructible. We begin with two points, which we will call p_0 and p_1, and we define the distance between the two points, $|p_0p_1|$, to be 1. Note that the points can be either arbitrarily placed before beginning the construction, or they can simply be two corners of the sheet of paper.

- We say that a line l is constructible if we can form a fold along the line l.
- We say that a point p is constructible if we can construct two lines that cross at the point p.
- We say that a number α is constructible if we can construct two points a distance α apart.

Note that the number 1 was defined at the start by the distance between the two initial points.

Single Fold Origami Axioms. We shall begin with the most basic form of origami — that of making single folds. This means that we are only permitted to perform one fold at a time, and that we must unfold the paper before performing a second fold. Under these constraints there are seven "Origami Axioms," seven functions we can perform that produce a fold [Lang, 2010, pg 42-43]:

1. Given two points p_1 and p_2 we can fold a line that passes through them.
2. Given two points p_1 and p_2 we can fold a line that places point p_1 on point p_2.
3. Given two lines l_1 and l_2 we can make a fold that places line l_1 onto line l_2.
4. Given a point p_1 and a line l_1 we can make a fold perpendicular to line l_1 that passes through point p_1.
5. Given two points p_1 and p_2 and a line l_1 we can make a fold that places point p_1 onto line l_1 that passes through point p_2.
6. Given two points p_1 and p_2 and two lines l_1 and l_2 we can make a fold that simultaneously places point p_1 onto line l_1 and places point p_2 onto line l_2.
7. Given a point p_1 and two lines l_1 and l_2 we can make a fold perpendicular to line l_2 that places point p_1 on line l_1.

Building the cartesian plane. It would be helpful if we could describe points by their cartesian coordinates in $\mathbb{R}^2$, but to do this we have to first establish that we can form a set of orthonormal basis vectors — and that we can extend them such that we can sensibly talk about a point with coordinates $(4, 3)$, for example. Recall that we started with two points, p_0 and p_1, and that the distance between them is defined to be 1.

We can take the vector extending from p_0 to p_1 as the first basis vector e_1. First we would like to find our second basis vector, e_2, and to do this we will need to construct a line segment with magnitude equal to that of e_1 that extends from point p_0 at a right angle to the first basis vector e_1. Let us call the endpoint of this line segment p_1' (to emphasize that the line segment is one unit long, and is orthogonal to e_1). Then the second basis vector, e_2, will be the vector extending from point p_0 to point p_1'.

Function 1. *Given two points p_0 and p_1, construct a third point p_1' a distance $|p_0p_1|$ from point p_0, such that $\overline{p_0p_1} \perp \overline{p_0p_1'}$.*

- ▷ *Using Axiom 1, construct the line l_1 that passes through the two initial points p_0 and p_1.*
- ▷ *Using Axiom 4, construct the line l_2 perpendicular to l_1 and passing through the point p_1.*
- ▷ *Using Axiom 4, construct the line l_3 perpendicular to l_1 and passing through the point p_0.*
- ▷ *Using Axiom 3, construct the line l_4 that places line l_1 onto line l_2.*

The point just constructed, where lines l_4 and l_3 cross, is the point p_1'.

Proof. The line $\overline{p_0p_1}$ is perpendicular to the line $\overline{p_0p_1'}$ as required since point p_1' lies on line l_3, which is perpendicular to the line l_1 by definition, which is constructible using Axiom 4.

Observe that when forming line l_4 we had to place line l_1 onto line l_2, thus the acute angle formed by lines l_1 and l_4 must be equal to that formed by l_2 and l_4, therefore the angle formed by lines l_1 and l_2 is bisected by line l_4.

Since lines l_2 and l_3 are parallel, they are both perpendicular to the common line l_1, we know that the acute angle formed by l_2 and l_4 is equal to the acute angle formed by l_4 and l_3 (by alternate interior angles). Further, since angle $\angle p_0p_1p_1' \cong \angle p_0p_1'p_1$, the triangle $\triangle p_1p_0p_1'$ is an isosceles triangle with $|\overline{p_0p_1}| = |\overline{p_0p_1'}|$. □

We can form the second basis vector e_2 by inputting our initial points p_0 and p_1 to Function 1, and using the vector extending from our initial point p_0 to the output point p_1'.

Next, we need to construct the integers along the e_1 and e_2 axes. Note that if we are able to construct a point p_2 collinear with the given points p_0 and p_1 such that $|\overline{p_0p_1}| = |\overline{p_1p_2}|$, then by repeating this process using p_k and p_{k+1} we will be able to build up the integers along the number line to any desired integer.

Note that if we can extend the number line in one direction we will be able to extend it in the other direction, by symmetry. Finally, since we can get a point one unit away from point p_0 in the direction of e_2 from points p_0 and p_1 using Function 1, we will be able to use the same procedure used to extend the e_1-axis using the points p_0 and p_1 to extend the e_2-axis using the points p_0 and the point p_1' produced by Function 1 with the initial points p_0 and p_1.

Function 2. *Given two points p_0 and p_1 we can construct a third point p_2 that is collinear with the given points, such that $|\overline{p_0p_1}| = |\overline{p_1p_2}|$.*

- ▷ *Use Function 1 to produce point p_1' using points p_0 and p_1, such that $\angle p_1p_0p_1'$ is a right angle. Note that lines l_1 (through p_0 and p_1), l_2 (through p_1 and $\perp l_1$), and l_3 (through p_0 and $\perp l_1$) have been formed in the process of executing Function 1.*
- ▷ *Use Axiom 4 to form line $l_5 \perp l_3$ through point p_1'.*
- ▷ *Use Axiom 3 to form the line l_6 that places line l_5 on l_2.*

The point just constructed, where lines l_6 and l_1 cross, is the point p_2.

> *Proof.* Point p_2 is clearly collinear with points p_0 and p_1 since it lies on the same line, l_1, as do points p_0 and p_1. Also, $|p_0p_1| = |p_1p_2|$ by congruent triangles $\triangle p_1p_0p_1' \cong \triangle p_2p_1p_3$ (all three angles are equal, as is one side), where point p_3 is the intersection of lines l_5 and l_2. □

Since we can now find all the integers, positive and negative, along the e_1 and e_2 axes, we can now construct any point in $\mathbb{Z} \times \mathbb{Z}$ by simply extending a perpendicular from an integer constructed on each axis and finding their intersection.

Moving from $\mathbb{Z}\times\mathbb{Z}$ to $\mathbb{Q}\times\mathbb{Q}$. I know that two numbers a and b can be divided with a straightedge and compass using constructed parallel lines, as in [Judson, 2011, pg 301] and in [Dummit and Foote, 2004, pg 532]. The problem is that we currently can only measure distances along the axes, while the triangles we have to construct require that we be able to measure diagonally. This takes some planning, but is not very difficult once you know how to double an angle.

Function 3. *Given two constructible numbers α and β, we can construct their ratio $\frac{\alpha}{\beta}$.*

- ▷ *Construct point p_a a distance α along the e_1-axis, and a point p_b a distance β along the e_2-axis, both extending from the point p_0.*
- ▷ *Use Axiom 4 to erect a line $l_1 \perp \overline{p_0p_a}$ through point p_a.*
- ▷ *Use Axiom 5 to form the line l_2 that passes through point p_0 and places point p_b onto line l_1.*
- ▷ *Use Axiom 4 to construct the line l_3 perpendicular to the line l_2, passing through point p_b. Name the intersection of lines l_3 and l_1 point p_2. Note that $|\overline{p_0p_2}| = \beta$.*
- ▷ *Use Axiom 1 to construct line l_4 which passes through points p_0 and p_2.*
- ▷ *Use Function 1 to construct point p_1' one unit along the e_2 axis.*
- ▷ *Use Axiom 5 to form the line l_5 that passes through point p_0 and places point p_1' onto line l_4.*
- ▷ *Use Axiom 4 to construct the line $l_6 \perp l_5$ that passes through the point p_1'. Name the intersection of lines l_6 and l_4 point p_3. Note that $|\overline{p_0p_3}| = 1$.*

- ▷ *Use Axiom 1 to construct the line l_0 through the initial points p_0 and p_1.*
- ▷ *Use Axiom 4 to form the line $l_7 \perp l_0$ that passes through point p_3.*
  Name the intersection of lines l_7 and l_0 point p_r.

The length of the line segment is equal to the desired ratio — that is, $|\overline{p_0p_r}| = \frac{\alpha}{\beta}$.

(Note that the construction of this ratio on the e_2 axis is entirely similar, with the axes in the instructions reversed.)

Proof. To prove that $|\overline{p_0p_2}| = \beta$, we will have to start by naming the intersection of lines l_2 and l_3 as point p_x. Then the length of the line segments $\overline{p_bp_x}$ and $\overline{p_xp_2}$ must have equal length since they can be superimposed upon each other. Since side $\overline{p_0p_x}$ is shared by both $\triangle p_0p_xp_b$ and $\triangle p_0p_xp_2$, the triangles must be congruent (side-angle-side congruence). Similarly, it can easily be shown that $|\overline{p_0p_3}| = 1$.

Note that $\triangle p_0p_rp_3 \sim \triangle p_0p_ap_2$ — that is, both right triangles that share an angle, thus all three corresponding angles are equal, thus the triangles are similar.

Therefore, the ratio of corresponding sides must be equal: $\frac{|\overline{p_0p_a}|}{|\overline{p_0p_2}|} = \frac{|\overline{p_0p_r}|}{|\overline{p_0p_3}|}$.

Since we know $|\overline{p_0p_a}| = \alpha$, $|\overline{p_0p_2}| = \beta$, and $|\overline{p_0p_3}| = 1$, we can substitute, finding that $\frac{\alpha}{\beta} = \frac{|\overline{p_0p_r}|}{1} = |\overline{p_0p_r}|$. □

Phew, we now have the ability to construct any point in $\mathbb{Q}^2$ by simply extending a perpendicular from a ratio of integers constructed on each axis and finding their intersection.

The constructible numbers form a field. We saw in the last section that *any* two constructible numbers can be divided (though at the time, we only knew how to construct the integers). Now that we know we can construct the rationals, we suspect that the numbers constructible by origami may form a field — certainly all that we currently know how to construct forms a field (since the rationals form a field), but we would like to be sure that the field structure is maintained even if we discover non-rational numbers that can also be constructed.

To do this, we need to determine whether or not the origami constructible numbers are closed under addition, subtraction, and multiplication (we already handled division).

Function 4. *Given two constructible numbers α and β, we can construct their sum $\alpha + \beta$ or their difference $\alpha - \beta$.*

- ▷ *Construct p_a a distance α along the e_1-axis, extending from the point p_0.*
- ▷ *Construct p_b a distance β along the e_1-axis, extending from the point p_a.*
  Note: extend in the same direction as e_1 for addition, and in the opposite direction to e_1 for subtraction.

The length $|\overline{p_0p_b}| = \alpha + \beta$ (or $\alpha - \beta$ if subtracting).

Proof. It is obvious that the appropriate sum (or difference) has been constructed — but only if you can indeed construct any previously constructible points based from point p_a, rather than from p_0 with the aid of point p_1 as we have been.

The only real difficulty, then, is that we do not necessarily have a unit length based from point p_a to work with.

To obtain the needed unit length based from point p_a, we will need our axes — so use Axiom 1 to construct the line $l_1 = \overline{p_0p_a}$, and again to construct the line $l_2 = \overline{p_0p'_1}$ (where p'_1 is the output from Function 1, as usual). Note that $l_1 \perp l_2$. Now use Axiom 4 to construct the line $l_3 \perp l_2$ through point p'_1, and again to construct the line $l_4 \perp l_1$ through the point p_a. Name the intersection of lines l_3 and l_4 point $p_{a'}$. Finally, use Axiom 3 to form line l_5 that places line l_3 onto line l_4. Call the intersection of line l_5 and l_1 point p_2.

Since the lines l_3 and l_1 are one unit apart and the triangle formed is isosceles (as in the proof of Function 2), the point p_2 must be one unit away from the point p_a. Thus, the number β which was constructible from point p_0 using point p_1 is now constructible using point p_a and point p_2. □

So now we have established that the set of constructible numbers are closed under addition, subtraction, and division — and multiplication will be easy, given division.

Function 5. *Given two constructible numbers α and β, we can construct their product $\alpha\beta$.*

- ▷ *Construct the point p'_1 a distance one along the e_2-axis, and point p_b a distance β along the e_1-axis.*
- ▷ *Construct the point $p_{b'}$ a distance $\frac{1}{\beta}$ along the e_2-axis using Function 3.*
- ▷ *Construct the point p_a a distance α along the e_1-axis.*
- ▷ *Construct the point p_{ab} a distance $\frac{\alpha}{\frac{1}{\beta}} = \alpha\beta$ along the e_1-axis using Function 3.*

So all we had to do to multiply α and β is divide α by the reciprocal of β.

Proof. We used Function 3 to divide one by β, getting us the ratio $\frac{1}{\beta}$. We then used Function 3 again to divide α by the ratio $\frac{1}{\beta}$ we just found, and after simplifying this complex fraction we find that the result is equal to $\alpha\beta$. □

Since we now know that the set of constructible numbers is closed under addition, subtraction, multiplication, and division, we can conclude that the set of constructible numbers form a field.

The field of constructible numbers is closed under taking square roots. Now that we have a field of rational numbers we would like to see if there are other numbers we can adjoin to it using origami. Taking as inspiration the methods used in [Judson, 2011, pg 301] and [Dummit and Foote, 2004, pg 532], we immediately see a method of constructing the square root of any constructible number — which would mean that the field of origami constructible numbers contains the entire field of straightedge and compass constructible numbers.

Function 6. *Given a constructible number α, the number $\sqrt{\alpha}$ is also constructible.*

- ▷ *Construct the point p_a a distance α in the negative e_2 direction, extending from point p_0.*
- ▷ *Construct the point p'_1 using Function 1 (in the positive e_2 direction).*
- ▷ *Use Axiom 1 to make the line l_1 that passes through points p_0 and p_1.*
- ▷ *Use Axiom 1 to make the line l_2 that passes through the points p_0 and p'_1.*
- ▷ *Use Axiom 2 to make the line l_3 that places point p'_1 onto point p_a.*
  Call the intersection of lines l_2 and l_3 the point p_c.
- ▷ *Use Axiom 5 to make the line l_4 that passes through point p_c and places point p'_1 onto line l_1.*
- ▷ *Use Axiom 4 to make the line $l_5 \perp l_4$ that passes through point p'_1.*
  Call the intersection of lines l_5 and l_1 the point p_r.

The length of the line segment $\overline{p_0p_r}$ is equal to $\sqrt{\alpha}$.

Proof. First, we note that the triangle $\triangle p_a p_r p'_1$ is a right triangle with the angle $\angle p_a p_r p'_1$ a right angle, since the line segment $\overline{p'_1 p_a}$ forms the diameter of a circle centered at p_c with radius $|\overline{p_c p'_1}|$. Put another way, since the three points p'_1, p_r, and p_a are all a distance $|\overline{p_c p'_1}|$ from the point p_c, the three points p'_1, p_r, and p_a all lie on a common circle centered at point p_c (though we did not draw the circle, since we do not get to use a compass in origami constructions).

Let $\angle p_0 p'_1 p_r = \theta$ and $\angle p_0 p_r p'_1 = \phi$. Then θ and ϕ are complementary angles, since the sum of angles in a triangle must add to π radians and the right angle accounts for $\frac{\pi}{2}$ radians — so $\theta + \phi = \frac{\pi}{2}$.

Since the angle $\angle p'_1 p_r p_a$ is a right angle, and the angle $\angle p_0 p_r p_a$ is the complement to ϕ, we know that the angle $\angle p_0 p_r p_a = \theta$. Thus, the angle $\angle p_0 p_a p_r = \phi$ since it is complemented by θ.

Therefore $\triangle p'_1 p_0 p_r \sim \triangle p_r p_0 p_a$ (all three corresponding angles are equal).
Similar triangles have equal side length ratios, thus $\frac{|\overline{p_0 p'_1}|}{|\overline{p_0 p_r}|} = \frac{|\overline{p_0 p_r}|}{|\overline{p_0 p_a}|}$.
Rearranging we find that $|\overline{p_0 p'_1}| \cdot |\overline{p_0 p_a}| = |\overline{p_0 p_r}| \cdot |\overline{p_0 p_r}| = |\overline{p_0 p_r}|^2$.
Recognizing that $|\overline{p_0 p'_1}| = 1$ and $|\overline{p_0 p_a}| = \alpha$, we can substitute — when we do this, we find that $1 \cdot \alpha = \alpha = |\overline{p_0 p_r}|^2$.

By taking square roots on both sides of this equation we obtain the desired result: $\sqrt{\alpha} = |\overline{p_0 p_r}|$. □

We now have the ability to construct any number using origami that could be constructed using a straightedge and compass. Nice.

The field of constructible numbers is closed under taking cube roots. Now that our origami constructions are as powerful as the classical straightedge and compass constructions, lets kick it up a notch — lets do something new with origami. Taking inspiration from the solution to the general cubic in [Koshiro,], we find that we can take the cube root of any constructible number.

Function 7. *Given a constructible number α, the number $\sqrt[3]{\alpha}$ is also constructible.*

- ▷ *Use Axiom 1 to construct line l_1 through points p_0 and p_1.*
- ▷ *Use Function 1 to construct the point p_1'.*
- ▷ *Use Axiom 1 to construct the line l_2 through points p_0 and p_1'.*
- ▷ *Construct the point p_a a distance α along the e_2-axis from point p_0.*
- ▷ *Construct the point p_{-a} a distance α along the negative e_2-axis from point p_0.*
- ▷ *Construct the point p_{-1} a distance one along the negative e_1-axis from point p_0.*
- ▷ *Use Axiom 4 to make the line $l_3 \perp l_1$ that passes through point p_{-1}.*
- ▷ *Use Axiom 4 to make the line $l_4 \perp l_2$ that passes through point p_a.*
- ▷ *Use Axiom 6 to make the line l_5 that simultaneously places point p_{-a} on line l_4 and point p_1 onto line l_3.*
  Name the point of intersection of lines l_5 and l_1 as point p_2, and the point of intersection of the lines l_5 and l_2 as point p_3.
- ▷ *Use Function 3 to find the ratio of $|\overline{p_0p_2}|$ to $|\overline{p_0p_3}|$.*

The ratio $\frac{|\overline{p_0p_2}|}{|\overline{p_0p_3}|} = \sqrt[3]{\alpha}$.

Proof. To start, let us observe that the coordinates of point p_{-a} is $(0, -\alpha)$, and of point p_1 is $(1, 0)$. Also, the equation of the line l_3 is $x + 1 = 0$, and of line l_4 is $y - \alpha = 0$. Finally, we need the equation of line l_5, so we will parameterize it as $y = mx + u$.

Let parabola P_1 be the parabola with focus p_1 and directrix l_3. Then the equation for P_1 is $y^2 = 4x$. The line l_5 must be tangent to parabola P_1 at some point, call this point (x_1, y_1). Using implicit differentiation, we find the derivative of the parabola at the point (x_1, y_1) to be $m = \frac{4}{2y_1} = \frac{2}{y_1}$. Then the equation of the tangent line must be $y - y_1 = m(x - x_1) = \frac{2}{y_1}(x - x_1)$ or equivalently, $y = \frac{2}{y_1}x - \frac{2x_1}{y_1} + y_1$.

Therefore the parameter values must be $m = \frac{2}{y_1}$ and $u = -\frac{2x_1}{y_1} + y_1$, or $u = -x_1 m + \frac{2}{m}$. Since the point (x_1, y_1) must also lie on the parabola, it must also be the case that $y_1^2 = 4x_1$, therefore $x_1 = \frac{y_1^2}{4} = \frac{\frac{4}{m^2}}{4} = \frac{1}{m^2}$. Thus, $u = -m\frac{1}{m^2} + \frac{2}{m} = -\frac{1}{m} + \frac{2}{m} = \frac{1}{m}$.

Let parabola P_2 be the parabola with focus p_{-a} and directrix l_4. Then the equation for P_2 is $x^2 = -4\alpha y$. Since the line l_5 must also be tangent to parabola P_2 at some point, we will call this point (x_2, y_2). Using implicit differentiation, we find the derivative of P_2 at point (x_2, y_2) to be $\frac{2x_2}{-4\alpha} = \frac{-x_2}{2\alpha} = m$. The equation of the tangent line must then be $y = \frac{-x_2}{2\alpha}x + \frac{x_2^2}{2\alpha} + y_2$.

Therefore the parameter values must be $m = \frac{-x_2}{2\alpha}$ and $u = y_2 + \frac{x_2^2}{2\alpha}$ or, subsituting the value of x_2, $u = y_2 + 2\alpha m^2$. Since the point (x_2, y_2) lies on P_2 it must be the case that $x_2^2 = -4\alpha y_2$, or rather $y_2 = \frac{x_2^2}{-4\alpha} = -\alpha m^2$. Substituting, we find that $u = -\alpha m^2 + 2\alpha m^2 = \alpha m^2$.

Bringing these equations together, we first note that investigating parabola P_1 gave us the equation for line l_5 as $y = mx + \frac{1}{m}$ and parabola P_2 gave us the equation for line l_5 as $y = mx + \alpha m^2$. Setting these forms of the equation equal to each other we find that $mx + \frac{1}{m} = mx + \alpha m^2$ or equivalently, $\frac{1}{m} = \alpha m^2$.

Rearranging this we find $\frac{1}{m^3} - \alpha = 0$ or if we define t as the reciprocal of the slope of the line l_5, we get $t^3 - \alpha = 0$. Observing that $t = \frac{|\overline{p_0p_2}|}{|\overline{p_0p_3}|}$, it is clear that $t = \sqrt[3]{\alpha}$. □

So the field of numbers constructible by single fold origami is closed under the taking of square roots, as is the field of numbers constructible by straightedge and compass, but the origami constructible numbers are also closed under the taking of cube roots.

Conclusion. We determined that the origami constructible numbers form a field closed under taking square and cube roots — but what does this buy us?

Recalling the classical straightedge and compass problems of doubling the cube and trisecting an angle, which were eventually found to be unsolvable, we observe that these are easily solvable using origami. The problem of doubling the cube can be reduced to constructing the cube root of two, which since the origami constructible numbers are closed under taking cube roots can clearly be done. The problem of trisecting an angle can be reduced to solving a cubic equation — and the general cubic can be solved in a variety of ways using single fold origami [Koshiro,] [Hull, 2011].

Exploring what more can be accomplished using origami, such as solving higher degree equations using folds other than the single fold Axioms presented in this paper [Lang, 2004], will have to be reserved for a future paper.

References

[Dummit and Foote, 2004] Dummit, D. S. and Foote, R. M. (2004). *Abstract Algebra*. 3rd edition.

[Hull, 2011] Hull, T. C. (2011). Solving cubics with creases: The work of Beloch and Lill. *The American Mathematical Monthly*, 118(10):954–964.

[Judson, 2011] Judson, T. W. (2011). *Abstract Algebra: Theory and Applications*.

[Koshiro,] Koshiro, H. Origami construction. `http://origami.ousaan.com/library/conste.html`.

[Lang, 2004] Lang, R. J. (2004). Angle quintisection. `http://www.langorigami.com/science/math/quintisection/quintisection.pdf`.

[Lang, 2010] Lang, R. J. (2010). Origami and geometric constructions. `http://www.langorigami.com/science/math/hja/origami_constructions.pdf`.