Math 290 Name: Key Dr. Beezer
Exam 2 Spring 2014
Chapter V

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to create and row-reduce matrices.

1. Determine if the set of column vectors, T, is linearly independent or not, including an accurate justification for your answer. (15 points)

$$T = \left\{ \begin{bmatrix} -1 \\ -3 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 4 \\ 3 \\ -2 \end{bmatrix} \right\}$$

According to theorem LIVRN we can start by making a matrix with these vectors as columns and "row-reducing"

$$A = \begin{bmatrix} -1 & 1 & 4 \\ -3 & 2 & 7 \\ 0 & 1 & 4 \\ -2 & 2 & 3 \\ -1 & -1 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we see that $r = 3 = n$, so by theorem LIVRN, T is a linearly independent set.

2. Determine if the vector $\mathbf{w}$ is in the set $U = \langle T \rangle$. (15 points)

$$\mathbf{w} = \begin{bmatrix} 2 \\ -2 \\ -4 \\ -1 \end{bmatrix} \qquad T = \left\{ \begin{bmatrix} 1 \\ -4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 3 \\ -1 \end{bmatrix} \right\}$$

By the definition of a span, we want to know if there are scalars a_1, a_2, a_3 so that $a_1 \begin{bmatrix} 1 \\ -4 \\ 2 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 5 \\ -4 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ -7 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \\ -1 \end{bmatrix}$.

By SLSLC, a_1, a_2, a_3 are a solution to a system w/ augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ -4 & 5 & -7 & -2 \\ 2 & -4 & 3 & -4 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

to solve system

Last column is a pivot column so by theorem RCLS, the system has no solution.

Hence, $\underset{\sim}{y} \notin \langle T \rangle$.

3. Find a linearly independent set R whose span is the null space of A (in other words, $\langle R\rangle = \mathcal{N}(A)$). Explain how you know your answer has the required properties. (20 points)

$$A = \begin{bmatrix} -3 & 1 & -5 & 5 & 3 \\ -4 & 1 & -7 & 6 & 5 \\ -4 & 1 & -7 & 6 & 5 \end{bmatrix}$$

This is a straight application of Theorem BNS. We need a row-reduced version of A.

$$A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Analysis: $F = \{3, 4, 5\}$

white chalk entries

chalk colored entries

$$R = \left\{ \begin{bmatrix} \\ \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

slots 3, 4, 5 owing to free variables x_3, x_4, x_5 in homogenous system $LS(A, \mathbf{0})$

Theorem BNS gives span & linear independence.

4. Find a linearly independent set T with the same span as S (in other words $\langle T\rangle = \langle S\rangle$). (20 points)

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \end{bmatrix} \right\}$$

Apply Theorem BS. Create a matrix whose columns are the vectors of S, and row-reduce.

$$B = \begin{bmatrix} 1 & -1 & -2 & -3 & 4 \\ 1 & 0 & -5 & 1 & 0 \\ 0 & 1 & -3 & 5 & -5 \\ -1 & -1 & 8 & -1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Analysis: $D = \{1, 2, 4\}$

T is the columns numbered 1, 2 & 4 from B:

$$T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 5 \\ -1 \end{bmatrix} \right\}$$

5. Suppose that $\mathbf{u} \in \mathbb{C}^n$ is a vector. Prove that $1\mathbf{u} = \mathbf{u}$. Provide reasons for each deduction and employ our indexing notation for entries of vectors. (Do not simply quote this as a result from Theorem VSPCV.) (15 points)

For $1 \le i \le n$,

scalar equality

$$[1\mathbf{u}]_i = 1[\mathbf{u}]_i \quad \text{Definition CVSM}$$

$$= [\mathbf{u}]_i \quad \text{Property OCN}$$

So by Definition CVE, $1\mathbf{u} = \mathbf{u}$.

vector equality

6. Suppose that $A = [\mathbf{A}_1|\mathbf{A}_2|\mathbf{A}_3|\dots|\mathbf{A}_n]$ is a matrix and that both $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ are solutions to $\mathcal{LS}(A, b)$. Prove that $\mathbf{x} - \mathbf{y}$ is a solution to the homgeneous system $\mathcal{LS}(A, \mathbf{0})$. (You may assume that vector subtraction is defined by $[\mathbf{x} - \mathbf{y}]_i = [\mathbf{x}]_i - [\mathbf{y}]_i$, for $1 \le i \le n$.) (15 points)

Consider $[\mathbf{x}-\mathbf{y}]_1 A_1 + [\mathbf{x}-\mathbf{y}]_2 A_2 + \dots + [\mathbf{x}-\mathbf{y}]_n A_n$

$= ([\mathbf{x}]_1 - [\mathbf{y}]_1)A_1 + ([\mathbf{x}]_2 - [\mathbf{y}]_2)A_2 + \dots + ([\mathbf{x}]_n - [\mathbf{y}]_n)A_n$ vector subtraction

$= [\mathbf{x}]_1 A_1 - [\mathbf{y}]_1 A_1 + [\mathbf{x}]_2 A_2 - [\mathbf{y}]_2 A_2 + \dots + [\mathbf{x}]_n A_n - [\mathbf{y}]_n A_n$ Property DSAC

$= ([\mathbf{x}]_1 A_1 + [\mathbf{x}]_2 A_2 + \dots + [\mathbf{x}]_n A_n) - ([\mathbf{y}]_1 A_1 + [\mathbf{y}]_2 A_2 + \dots + [\mathbf{y}]_n A_n)$ Property CC

$= \mathbf{b} - \mathbf{b}$ $\quad \mathbf{x}, \mathbf{y}$ solutions to $LS(A, \mathbf{b})$ & Theorem SLSLC

$= \mathbf{0}$ $\quad$ Property AIC.

By SLSLC this says $\mathbf{x} - \mathbf{y}$ is a solution to $LS(A, \mathbf{0})$