Math 290
Exam 6
Chapter LT

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to row-reduce matrices and include the output in your answer as justification.

1. Consider the linear transformation $T$ below. $P_1$ is the vector space of polynomials with degree at most 1, and $M_{22}$ is the vector space of $2\times 2$ matrices. (45 points)

$$T\colon M_{22} \to P_1,\quad T\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right) = (a+2b+3c-3d) + (3a-b+2c+5d)x$$

(a) Compute the kernel of $T$, $\mathcal{K}(T)$.

$T\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right) = \mathbf{0} = 0+0x \Rightarrow \begin{matrix} a+2b+3c-3d=0 \\ 3a-b+2c+5d=0\end{matrix}$

augmented matrix $\xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0\end{array}\right] \Rightarrow \begin{matrix} a=-c-d \\ b=-c+2d\end{matrix} \Rightarrow \begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} -c-d & -c+2d \\ c & d\end{bmatrix} =$

$= c\begin{bmatrix} -1 & -1 \\ 1 & 0\end{bmatrix} + d\begin{bmatrix} -1 & 2 \\ 0 & 1\end{bmatrix}$ So $K(T) = \left\langle\left\{\begin{bmatrix} -1 & -1 \\ 1 & 0\end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 1\end{bmatrix}\right\}\right\rangle$

(b) Is $T$ injective? Why? If not, find two elements of the domain that $T$ takes to the same element of the codomain.

No, Theorem KILT + $K(T) \neq \{\mathbf{0}\}$

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}\right) = 0+0x = T\left(\begin{bmatrix} -1 & -1 \\ 1 & 0\end{bmatrix}\right)$

Thm LTTZZ — in kernel, see above

(c) Compute the range of $T$, $\mathcal{R}(T)$.

$4 = \dim(M_{22}) = n(T) + r(T) = 2 + r(T) \Rightarrow r(T) = 2$

So the range of $T$ is a subspace of $P_1$ (which has dimension 2) that has dimension 2. So $R(T) = P_1$

(d) Is $T$ surjective? Why? If not, find an element of the codomain with an empty preimage.

Yes, by Theorem RSLT, since $R(T) = P_1$.

(e) If $T$ invertible? Why?

No. Theorem ILTIS <u>requires</u> $T$ to be <u>both</u> injective and surjective ($T$ is not both). <u>Or</u> $\dim(M_{22}) = 4 \neq 2 = \dim(P_1)$ and apply Theorem IVSED.

2. Consider the linear transformation $S$ below, which is invertible (you may assume this). Find a formula for the outputs of the inverse linear transformation $S^{-1}$. $P_1$ is the vector space of polynomials with degree at most 1. (25 points)

$$S\colon P_1 \to \mathbb{C}^2, \quad S(a+bx) = \begin{bmatrix} 2a+5b \\ a+3b \end{bmatrix}$$

Compute pre-images of a basis for the codomain: $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

$S^{-1}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$: $S(a+bx) = \begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \begin{matrix}2a+5b=1\\a+3b=0\end{matrix} \Rightarrow \begin{matrix}a=3\\b=-1\end{matrix}$

$S^{-1}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \{3-x\} \Rightarrow S^{-1}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 3-x$

$S^{-1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$: $S(a+bx) = \begin{bmatrix}0\\1\end{bmatrix} \Rightarrow \begin{matrix}2a+5b=0\\a+3b=1\end{matrix} \Rightarrow \begin{matrix}a=-5\\b=2\end{matrix}$

$S^{-1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \{-5+2x\} \Rightarrow S^{-1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = -5+2x$

Then,

$$\begin{aligned} S^{-1}\left(\begin{bmatrix}r\\s\end{bmatrix}\right) &= S^{-1}\left(r\begin{bmatrix}1\\0\end{bmatrix} + s\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= r\,S^{-1}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + s\,S^{-1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= r(3-x) + s(-5+2x) \\ &= (3r-5s) + (-r+2s)x \end{aligned}$$

3. Prove that the function $T$ below is a linear transformation. $P_1$ is the vector space of polynomials with degree at most 1. (15 points)

$T\colon P_1 \to \mathbb{C}^2, \quad T(a+bx) = \begin{bmatrix} 3a-b \\ 4b \end{bmatrix}$

① $T((a_1+b_1x)+(a_2+b_2x)) = T((a_1+a_2)+(b_1+b_2)x)$

$= \begin{bmatrix} 3(a_1+a_2)-(b_1+b_2) \\ 4(b_1+b_2) \end{bmatrix} = \begin{bmatrix} (3a_1-b_1)+(3a_2-b_2) \\ 4b_1+4b_2 \end{bmatrix}$

$= \begin{bmatrix} 3a_1-b_1 \\ 4b_1 \end{bmatrix} + \begin{bmatrix} 3a_2-b_2 \\ 4b_2 \end{bmatrix} = T(a_1+b_1x)+T(a_2+b_2x)$

② $T(\alpha(a+bx)) = T(\alpha a+(\alpha b)x) = \begin{bmatrix} 3(\alpha a)-\alpha b \\ 4(\alpha b) \end{bmatrix}$

$= \begin{bmatrix} \alpha(3a-b) \\ \alpha(4b) \end{bmatrix} = \alpha\begin{bmatrix} 3a-b \\ 4b \end{bmatrix} = \alpha T(a+bx)$

4. Suppose that $S\colon U \to V$ is an invertible linear transformation. Then prove that $S^{-1}$ has one of the two defining properties of a linear transformation (either property, your choice, for full credit). (15 points)

Choose $\alpha \in \mathbb{C}$, $\mathbf{v} \in V$. Because $S$ is invertible, $S$ must be surjective (Theorem ILTIS) so there exists a $\mathbf{u} \in U$ with $S(\mathbf{u}) = \mathbf{v}$ & equivalently, $S^{-1}(\mathbf{v}) = \mathbf{u}$.

Then $S^{-1}(\alpha\mathbf{v}) = S^{-1}(\alpha S(\mathbf{u}))$

$= S^{-1}(S(\alpha\mathbf{u}))$ $\quad$ $S$ is a linear transformation

$= I_U(\alpha\mathbf{u})$

$= \alpha\mathbf{u}$

$= \alpha S^{-1}(\mathbf{v})$

which is the second defining property.