Math 290 Name: Key Dr. Beezer
Exam 7 Spring 2014
Chapter R

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use any Sage routines for matrix algebra as justification for your work. Sage routines for linear transformations may not be used.

1. Find the matrix representation of T, relative to the bases B and C, in other words, compute $M^T_{B,C}$. P_2 is the vector space of polynomials with degree at most 2. (15 points)

$$T\colon P_2 \to \mathbb{C}^2, \quad T\left(a+bx+cx^2\right) = \begin{bmatrix} 2a-b+c \\ -a+b-c \end{bmatrix}$$

$$B = \left\{1+2x, 1+3x+x^2, 1+7x+4x^2\right\} \quad C = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right\}$$

$$\rho_C(T(1+2x)) = \rho_C\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \rho_C\left(-1\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}$$

$$\rho_C(T(1+3x+x^2)) = \rho_C\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \quad '' \quad = \quad ''$$

$$\rho_C(T(1+7x+4x^2)) = \rho_C\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \rho_C\left(-5\begin{bmatrix}1\\2\end{bmatrix} + 4\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}-5\\4\end{bmatrix}$$

so $M^T_{B,C} = \begin{bmatrix} -1 & -1 & -5 \\ 1 & 1 & 4 \end{bmatrix}$

2. Consider the linear transformation S below, which is invertible (you may assume this). Use matrix reprsentations to find a formula for the outputs of the inverse linear transformation S^{-1}. No credit will be given for answers obtained by other methods. P_1 is the vector space of polynomials with degree at most 1 and M_{12} is the vector space of 1×2 matrices. (15 points)

$$S\colon P_1 \to M_{12}, \quad S\left(a+bx\right) = \begin{bmatrix} 5a+8b & -2a-3b \end{bmatrix}$$

Relative to $B = \{1, x\}$, $C = \{[1\ 0], [0\ 1]\}$ $\quad M^S_{B,C} = \begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}$

Then $M^{S^{-1}}_{C,B} = \left(M^S_{B,C}\right)^{-1} = \begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}$

$$S^{-1}([a\ b]) = \rho_B^{-1}\left(M^{S^{-1}}_{C,B}\,\rho_C([a\ b])\right) = \rho_B^{-1}\left(\begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\right)$$

$$= \rho_B^{-1}\left(\begin{bmatrix} -3a-8b \\ 2a+5b \end{bmatrix}\right) = (-3a-8b)(1) + (2a+5b)x$$

3. Consider the linear transformation T below. (35 points)

$$T\colon \mathbb{C}^2 \to \mathbb{C}^3, \quad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a+3b \\ a-b \\ -2a+b \end{bmatrix}$$

(a) Compute a matrix representation of T relative to the appropriate standard bases.

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

on sight

$$M^T_{B,C} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -2 & 1 \end{bmatrix}$$

(b) Use two change-of-basis matrices in the right way to compute a matrix representation of T relative to the bases X and Y. No credit will be given for answers obtained with other methods.

$$X = \left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \qquad Y = \left\{ \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$$M^T_{X,Y} = C_{C,Y}\, M^T_{B,C}\, C_{X,B}$$

$$C_{X,B} = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} \text{ on sight}$$

$$C_{C,Y} = C^{-1}_{Y,C} = \begin{bmatrix} -2 & 1 & -1 \\ -3 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 7 & -5 & 1 \\ 4 & -3 & 1 \end{bmatrix}$$

$$M^T_{X,Y} = \begin{bmatrix} 1 & -1 & 0 \\ 7 & -5 & 1 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -7 \\ 67 & -47 \\ 42 & -29 \end{bmatrix}$$

(c) Perform a crude check on your work by computing $T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$ by using the Fundamental Theorem of Matrix Representation twice, once with each of your two representations.

EZ: $$T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \rho_C^{-1}\left(M^T_{B,C}\, \rho_B\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)\right) = \rho_C^{-1}\left(\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$$

$$= \rho_C^{-1}\left(\begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix}\right) = 9\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-5)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix}$$

Work: $$T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \rho_Y^{-1}\left(M^T_{X,Y}\, \rho_X\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)\right) = \rho_Y^{-1}\left(\begin{bmatrix} 10 & -7 \\ 67 & -47 \\ 42 & -29 \end{bmatrix}\begin{bmatrix} -7 \\ -11 \end{bmatrix}\right)$$

$$= \rho_Y^{-1}\left(\begin{bmatrix} 7 \\ 48 \\ 25 \end{bmatrix}\right) = 7\begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix} + 48\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 25\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix}$$

4. Determine the eigenspaces of the linear transformation R below. P_2 is the vector space of polynomials with degree at most 2. (20 points)

$R\colon P_2 \to P_2,\quad R\left(a + bx + cx^2\right) = (-8a + 10b + 5c) + (-19a + 20b + 9c)\,x + (28a - 26b - 11c)\,x^2$

on sight, relative to $B = \{1, x, x^2\}$ $\quad M^{R}_{B,B} = \begin{bmatrix} -8 & 10 & 5 \\ -19 & 20 & 9 \\ 28 & -26 & -11 \end{bmatrix}$

Sage, eigenmatrix_right() $\quad \lambda = -3 \Rightarrow \underset{\sim}{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

$\lambda = 2 \Rightarrow \underset{\sim}{x} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ ← geometric multiplicity "only" 1

$\mathcal{E}_R(-3) = \langle\{1 + 2x - 3x^2\}\rangle$

$\mathcal{E}_R(2) = \langle\{x - 2x^2\}\rangle$

"un-coordinatize" e'vectors from Sage, relative to basis B

5. Suppose that $T\colon U \to U$ is a linear transformation and W is a subspace of U. Then we say W is T-invariant if for each $\mathbf{w} \in W$ we have $T(\mathbf{w}) \in W$. Choose exactly two of the three claims by circling the problem ((a), (b), (c)) and prove the claim. Full credit for two correct proofs. (Uncircled problems will not be graded, and if three problems are circled, the two lowest scores will be kept.) (15 points)

(a) The kernel of T, $\mathcal{K}(T)$, is T-invariant.

$\underset{\sim}{w} \in \mathcal{K}(T) \Rightarrow T(\underset{\sim}{w}) = \underset{\sim}{0} \in \mathcal{K}(T)$

$\underset{\sim}{0}$ is in every subspace

(b) For any eigenvalue λ of T, the eigenspace, $\mathcal{E}_T(\lambda)$, is T-invariant.

$\underset{\sim}{w} \in \mathcal{E}_T(\lambda) \Rightarrow T(\underset{\sim}{w}) = \lambda\underset{\sim}{w} \in \mathcal{E}_T(\lambda)$

subspaces are closed under scalar multiplication

(c) The range of T, $\mathcal{R}(T)$, is T-invariant.

$\underset{\sim}{w} \in \mathcal{R}(T) \Rightarrow T(\underset{\sim}{w}) \in \mathcal{R}(T)$ by <u>definition</u>