General Inner Product & Fourier Series

Advanced Topics in Linear Algebra, Spring 2014

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1 General Inner Product

The inner product is an algebraic operation that takes two vectors of equal length and computes a single number, a scalar. It introduces a geometric intuition for length and angles of vectors. The inner product is a generalization of the dot product which is the more familiar operation that's specific to the field of real numbers only. Euclidean space which is limited to 2 and 3 dimensions uses the dot product. The inner product is a structure that generalizes to vector spaces of any dimension. The utility of the inner product is very significant when proving theorems and runing computations. An important aspect that can be derived is the notion of convergence. Building on convergence we can move to represenations of functions, specifally periodic functions which show up frequently. The Fourier series is an expansion of periodic functions specific to sines and cosines. By the end of this paper we will be able to use a Fourier series to represent a wave function. To build toward this result many theorems are stated and only a few have proofs while some proofs are trivial and left for the reader to save time and space.

Definition 1.11 Real Inner Product
Let V be a real vector space and $a, b \in V$. An inner product on V is a function $\langle\ ,\ \rangle : V$ x $V \rightarrow R$ satisfying the following conditions:

(a) $\langle \alpha a + \alpha' b, c\rangle = \alpha\langle a, c\rangle + \alpha'\langle b, c\rangle$
(b) $\langle c, \alpha a + \alpha' b\rangle = \alpha\langle c, a\rangle + \alpha'\langle c, b\rangle$
(c) $\langle a, b\rangle = \langle b, a\rangle$
(d) $\langle a, a\rangle$is a positive real number for any $a \neq 0$

Definition 1.12 Complex Inner Product
Let V be a vector space. An inner product on V is a complex valued function $\langle\ ,\ \rangle : V$ x V $\rightarrow R$ satisfying the following conditions:

(a) $\langle \alpha a + \alpha' b, c\rangle = \alpha\langle a, c\rangle + \alpha'\langle b, c\rangle$
(b) $\langle c, \alpha a + \alpha' b\rangle = \alpha^*\langle c, a\rangle + \alpha'^*\langle c, b\rangle$
(c) $\langle a, b\rangle = \overline{\langle b, a\rangle}$
(d) $\langle a, a\rangle$is positive and real for all $a \in V$

An inner product of sesquilinear form on a complex vector space V is a map V x $V \rightarrow C$ that is linear in one argument and nonlinear in the other. The bilinear form which is linear in both arguments seperately is of a real vector space and is referred to as the dot product. The structure of the inner product offers insight into the geometric intuition which can be seen by the following theorems and definitions.

Definition 1.13 Norm
Let V be a vector space and $a \in V$. Then $||a|| = \langle a, a\rangle^{1/2}$ is the norm or length of any $a \in V$.

Theorem 1.11 Schwarz's Inequality
Let V be a vector space. Then $|\langle b, c\rangle| \leq ||b||\,||c||$ for all $b, c \in V$. With equality only when $b = \alpha c$ for some $\alpha \in F$

Definition 1.14 Orthogonal Vectors
If V is a vector space, $a, b \in V$ and $\langle a, b\rangle = 0$ then a and b are orthogonal to eachother.

Theorem 1.12 Parallelogram
If V is a vector space and $a, b \in V$ then,

$$||a + b||^2 + ||a - b||^2 = 2(||a||^2 + ||b||^2)$$

Also, a and b are orthogonal if and only if,

$$||a + b||^2 = ||a||^2 + ||b||^2$$

Definition 1.15 Distance
Let V be an inner product space. Then the distance between any two vectors $a, b \in V$ is defined by,

$$\mathrm{d}(a, b) = ||a - b||$$

Theorem 1.13 Distance Corollary
Let V be an inner product space. Then the distance between any two factors $a, b \in V$ satisfies,
(a) $\mathrm{d}(b, c) \geq 0$ for any two vectors or $\mathrm{d}(b, c) = 0$ If and only if $b = c$
(b) $\mathrm{d}(b, c) = \mathrm{d}(c, b)$ (symmetry)

Any nonempty set V with the function d:Vx$V \rightarrow R$ that satisifies the Theorem 1.13 is called a **metric space** and the function d is called a metric on V. Thus we can say that any inner product space is also a metric space under the given metric. The idea of an inner product and metric raises concernes about topoligical issues of convergence, so the following definition is offered.

Definition 1.16 Convergence
Suppose we have a sequence (number of ordered elements, like a set but order matters and the same element may show up more than once) of vectors (v_n) in an inner product space. Then the set converges to $v \in V$ if:

$$\lim_{n\rightarrow\infty} \mathrm{d}(v_n, v) = 0$$
or
$$\lim_{n\rightarrow\infty} ||v_n - v|| = 0$$

Convergence becomes a very important concept when considering closedness and closures, completeness, and continuity of linear functionals and operators. In the finite dimension these concepts are straight forward. Subspaces are closed, inner product spaces are complete and linear functionals and operators are continuous. Things are not as simple for the infinite dimensional case.

Inner product spaces can be broken into two spaces, Euclidean and Hilbert. A Euclidean space refers to a finite dimensional linear space with an inner product. Specifically it refers to the 2 and 3 dimensions over the reals which is always complete by virtue of the fact that it is finite dimensional. A Hilbert space is an infinite dimensional inner product space which is complete for the norm induced by the inner product. Hilbert space is very similar to Euclidean space but there are some ways in which the infinite dimensionality leads to subtle differences. Luckily for the infinite dimensional case the intuitition from the finite dimensional case can be carried over and made use of.

There are also two types of Hilbert spaces. The complete Hilbert space and the pre-Hilbert space. An incomplete space with an inner product is referred to as a pre-Hilbert space and the complete space as a Hilbert space. A complete space respects the norm and has limits that allow calculus techniques to be used. It is also intuitively one without any points missing from it which is analogous to a closed set. Having a Hilbert space leads to many good things such as orthogonal projections.

Definition 1.17 Hilbert Space
The complex inner product space H is a pre-Hilbert space if there exists a complex inner product on H such that $||v|| = \langle v, v\rangle^{1/2}$ for all $v \in H$. If the pre-Hilbert space is complete then it called a Hilbert space. To be complete H must be a metric space and respect the convergence definition.

Definition 1.18 Banach Space
A vector space B is a Banach space if it is a complete normed vector space. Thus it is equipped with a norm and is complete in respect to that norm such that any sequence $(b_n) \in B$ converges.

Theorem 1.14 Orthogonal Projections (Projection Theorem)
Let H be a Hilbert space and $M \subset H$ be a closed subspace. The orthogonal projection P_M is defined by the following conditions:
(a) P_M is linear
(b) $P_M^2 = P_M$ (projection)
(c) $P_M^* = P_M$ (self-adjoint)
(d) rank(P_M) $= M$ and kernel(P_M)$= M^\perp$

Theorem 1.15 Orthogonal Complement
Let V be a finite-dimensional inner product space and W be a subspace of V. Then $V = W \oplus W^\perp$

Definition 1.19 Dual Space
Let V be a vector space over F. The Dual of V, V^* is the vector space of all linear transfor-

mations $T : V \to F$. Elements of V^* are called linear functionals.

Theorem 1.16 Riesz Representation
Let V be a finite dimensional inner product space and $f \in V^*$ be a linear functional on V. Then there exists a unique vector $x \in V$ that satisfies:

$$f(v) = \langle v, x \rangle$$

for all $v \in V$

Proof. If f is the zero functional then we can say $x = 0$. So let us assume $f \neq 0$. We want x so that $\langle v, x \rangle = 0$ for all $v \in \text{kernel}(f)$. Thus we should look for an x in $\text{kernel}(f)^{\perp}$. If $\dim(V) = n$ then $\dim(\text{kernel}(f)) = n - 1$. Thus we can choose a vector $u \in \text{kernel}(f)^{\perp}$and have,

$$V = \langle u \rangle \oplus \text{kernel}(f)^{\perp}$$

Now we want to find an $r \in F$ such that,

$$f(v) = \langle v, ru \rangle$$

for all $v \in V$. Specifically, for $v = u$, we want,

$$f(u) = \langle u, ru \rangle = \overline{r} \langle u, u \rangle = \overline{r}$$

Now take $r = \overline{f(u)}$ and we get,

$$x = \overline{f(u)} u$$

Any vector $v \in V$ has the form $v = au + bw$, with $w \in \text{kernel}(f)$, and thus,

$$\begin{aligned} \langle v, x \rangle &= \langle v, \overline{f(u)} u \rangle \\ &= f(u) \langle v, u \rangle \\ &= f(u) a \\ &= f(au) \\ &= f(au + bw) \\ &= f(v) \end{aligned}$$

The Riesz Representation Theorem says that every continuous linear functional $f \in V^!$ arises in this way.That for any given element f of the dual space V^* there exists some $x \in V$ so that $f(v) = \langle v, x \rangle$. Thus you can think of linear functionals as elements of the Hilbert space, and vice versa. In a more familiar and mathematical term we say that the dual of V, V^*, is isomorphic to V.

2 Fourier Series

A Fourier series is an expansion of a periodic function in terms of an infinite sum of sines and cosines. A Fourier series uses the relationship of orthogonality between the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is useful as a way to break up an arbitrary periodic function into a set of simple terms. The simple terms can be plugged in and solved individually. Then it can be recombined to obtain the solution to the original problem or offer an approximation with an accuracy that is desired or practical.

2.1 Fourier Series and The Periodic Function on [-π, π]

Definition 1.21 Real form of Fourier Series of a periodic function
Suppose we have a periodic function $f(x)$ with a period of 2π, where $k = 0, 1, ...$ Then the series is defined as,

$$\mathrm{a}_o/2 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

where the coefficients a_k, b_k are defined as,

$$\mathrm{a}_k = 1/\pi \int_{-\pi}^{\pi} f(x) \cos kx dx,$$
$$\mathrm{b}_k = 1/\pi \int_{-\pi}^{\pi} f(x) \sin kx dx$$

Definition 1.22 Complex form of Fourier Series of a periodic function
Suppose we have a periodic function $f(x)$ with a period of 2π, where $k = 0, 1, ...$ Then the series is defined as,

$$\sum_{-\infty}^{\infty} c_k e^{ikx}$$

where c_k is defined as,

$$\mathrm{c}_k = 1/2\pi \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

Theorem 1.21 Fourier Expansion
Suppose V is a vector space and $B = \{u_1, ..., u_n\}$ is an orthonormal basis for V. Then for any $v \in V$,

$$v = \langle v, u_1 \rangle u_1 + ... + \langle v, u_n \rangle u_n$$

Where the $\langle v, u_i \rangle$ coordinates are the Fourier coefficients of v with respect to B and $\langle v, u_1 \rangle u_1 + ... + \langle v, u_n \rangle u_n$ is the Fourier expansion of v with respect to B.

Theorem 1.22 Bessel's Inequality
Suppose V is a vector space and $B = \{u_1, ..., u_n\}$ is an orthonormal basis for V. Then for any $v \in V$,

$$||v||^2 \geq |\langle v, u_1\rangle|^2 + ... + |\langle v, u_n\rangle|^2$$

When the right hand side converges.

Theorem 1.23 Inner Product of continuous functions over the complex numbers

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$$

The L^2 space consists of equivalence classes of functions. Two functions represent the same L^2 function if the difference in sets measures to zero. From Theorem 1.23 we get $\langle f, g\rangle$ as an inner product because if we had $f = 0$ then $\langle f, f\rangle = 0$. L^2 from $[-\pi, \pi]$ is a vector space with structure that makes ease of representations of functions that are periodical and consist of sines and cosines. Moving towards our goal to represent a wave function we have the following definitions and theorems.

Definition 1.23 L^2([-π, π] Space
L^2 is the set of all complex-valued functions on $[-\pi, \pi]$ that satisfy,

$$\int_{-\pi}^{\pi} |f(x)|^2 dx\langle\infty$$

Where the inner product is,

$$\langle f, g\rangle = 1/\pi \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$$

Theorem 1.24 Fourier Series of L^2
If $f \in L^2$ then its Fourier Series is,

$$a_o/2 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

Where

$$\text{a}_k = \langle f, \cos kt\rangle$$
$$\text{b}_k = \langle f, \sin kt\rangle$$

Theorem 1.25 Convergence in the mean (in the norm)
Let $f \in L^2$ and set,

$$\text{g(t)} = \text{a}_o/2 + \sum_{k=1}^{N}(a_k \cos kt + b_k \sin kt)$$

Then $g(t) \in L^2$ and,

$$||g(t) - f||^2 = 1/\pi \int_{-\pi}^{\pi} |g(t) - f(t)|^2 dt \to 0$$
as N$\to$ 0

Theorem 1.26 Inner Product Equivalence
Suppose V is an inner product space with orthonormal basis $B = \{u_1, ..., u_k\}$. Then any vector $v \in V$ has the following equivalences,

$$\text{v} = \sum_{k=1}^{N} \langle v, u_k \rangle u_k$$
and
$$||v||^2 = \sum_{k=1}^{N} |\langle v, u_k \rangle|^2$$

Intuitively these make sense because v is a linear combination of the vectors of B with some constants α_k and thus any two vectors of the basis have the inner products $\langle u_i, u_j \rangle = 0$ and $\langle u_i, u_i \rangle = 1$ for every value of i up to N. From this we can calculate that $\langle v, u_k \rangle = \alpha_k$

Definition 1.24 Closed System
If V is an inner product space and $B = \{u_1, ..., u_k\}$ is an orthonormal basis of V then $\{u_1, ..., u_k\}$ is a closed system if for every $v \in V$,

$$v = \sum_{k=1}^{\infty} \langle v, u_k \rangle u_k$$

That is if the sequence $\{v_N\}$ defined by,

$$v_N = \sum_{k=1}^{N} \langle v, u_k \rangle u_k$$

converges to v:

$$||v - v_N|| \to 0$$

as,

$$N \to \infty$$

Theorem 1.27 Orthonormal system closure
The orthonormal basis $\{u_1, ..., u_k\}$ of an inner product space V is closed if and only if,

$$||v||^2 = \sum_{k=1}^{\infty} |\langle v, u_k \rangle|^2$$

for every $v \in V$.

Proof.

$$||v||^2 = ||v_n||^2 - ||v - v_n||^2$$

We have,

$$||v_n||^2 = \sum_{k=1}^{N} |\langle v, u_k \rangle|^2$$

and by Bessel's inequality,

$$\lim_{n\to\infty} ||v_n||^2 = \sum_{k=1}^{\infty} |\langle v, u_k \rangle|^2$$

exists.
If the system is closed then,

$$||v - v_n|| \to 0$$

as

$$N \to 0$$

and,

$$\lim_{n\to\infty} ||v_n||^2 = ||v||^2$$

So,

$$||v||^2 = \sum_{k=1}^{\infty} |\langle v, u_k \rangle|^2$$

□

If you graph the square wave function you will notice that it is an odd function. Therefore the coefficients of the cosine terms will be zero. Since $L = \pi$, the coefficients of the sine terms can be computed with the following theorem.

Theorem 1.28 Parseval's
Let V be an inner product space and $B = \{u_1, ..., u_k\}$ be a closed orthonormal basis in V. Then for any $v, w \in V$,

$$\langle v, w \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}$$

Where,

$$a_k = \langle v, u_k \rangle, b_k = \langle w, u_k \rangle$$

And,

$$||v||^2 = \sum_{k=1}^{\infty} |\langle v, u_k \rangle|^2 = \sum_{k=1}^{\infty} |a_k|^2$$

Proof. We have

$$\begin{aligned} ||v-w||^2 &= \langle v-w, v-w\rangle \\ &= \langle v,v\rangle + \langle w,w\rangle - 2\langle v,w\rangle \\ &= ||v||^2 + ||w||^2 - 2\langle v,w\rangle \end{aligned}$$

where,

$$2\langle v,w\rangle = ||v-w||^2 - ||v||^2 - ||w||^2$$

and,

$$\begin{aligned} ||v-w||^2 &= \sum_{k=1}^{\infty} |a_k - b_k|^2 \\ &= \sum_{k=1}^{\infty} |a_k|^2 + \sum_{k=1}^{\infty} |b_k|^2 - 2\sum_{k=1}^{\infty} a_k b_k \\ &= ||v||^2 + ||w||^2 - 2\sum_{k=1}^{\infty} a_k b_k \end{aligned}$$

By comparing the two sets of equations, we obtain

$$\langle v,w\rangle = \textstyle\sum_{k=1}^{\infty} a_k b_k$$

□

We now have a good intuition of inner products and their use as well as resulting theorems that give us some tools for tackleing periodic functions. Our hammers will be the Fourier series and Parseval's theorem. We also have ideas such as convergence, completeness and closedness. Now let us move on to some application and see what these newfound tools can offer us in the face of a problem.

Example: The input to an electrical circuit that switches between a high and a low state with time period 2π can be represented by the boxcar function,

$$f(x) = 1 \text{ when } 0 \leq x \langle \pi$$
and
$$f(x) = -1 \text{ when } -\pi \leq x \langle 0$$

The periodic expansion of this function is referred to as the square wave function. Generally this is the input to an electrical circuit that switches from a high to low state with time period T which can be represented by the general square wave function with the basic period, (insert picture)

$$f(x) = 1 \text{ when } 0 \leq x \langle T/2$$
and
$$f(x) = -1 \text{ when } -T/2 \leq x \langle 0$$

$$\begin{aligned} b_k &= 1/\pi \int_{-\pi}^{\pi} f(x) \sin kx dx \\ &= 2/\pi \int_{0}^{\pi} f(x) \sin kx dx \\ &= -2/k\pi \quad \cos kx|_0^\pi \\ &= -2/k\pi((-1)^k - 1) \end{aligned}$$

Notice that $((-1)^k - 1) = 1 - 1 = 0$ if k is even $(2k)$ but $= -2$ if k is odd $(2k+1)$. Thus, $b_{2k} = 0$ and $b_{2k+1} = -2/k\pi(-2) = 4/(2k+1)\pi$ and we get,

$$\begin{aligned} f(x) &= 4/\pi \sum_{k=odd}^{\infty} (1/k) \sin kx \\ &= 4/\pi \sum_{k=0}^{\infty} (\sin(2k+1)x)/2k+1 \\ &= 4/\pi(\sin x + \sin 3x/3 + \sin 5x/5 + ...) \end{aligned}$$

For the representation of the general square wave we obtain $a_n = 0, b_{2k} = 0$ and thus,

$$b_{2k+1} = (4/(2k+1)\pi) \sin((2(2k+1)\pi x)/T)$$

and we get,

$$f(x) = 4/\pi \sum_{k=0}^{\infty} (1/2k+1) \sin((2(2k+1)\pi x)/T)$$

We started with the defining property of an inner product which is simply a scalar computed from two vectors. From this we derived geometric results such as norm,orthogonality, distance, and made our way to inner product spaces. Convergence made its appearance as well as the Riesz Representation Theorem as an opener. Moving from general structure we focused in on the specific vector space of L^2 and the represention from a Fourier series. Now we are able to take wave functions and represent them with accurate approximations. This paper offers a good foundation for those looking into represenations and the general inner product structure and function.

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