Polar Decomposition of a Matrix

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What is it?

Definition (Right Polar Decomposition)

The right polar decomposition of a matrix $A \in \mathbb{C}^{m \times n}$ $m \geq n$ has the form $A = UP$ where $U \in \mathbb{C}^{m \times n}$ is a matrix with orthonormal columns and $P \in \mathbb{C}^{n \times n}$ is positive semi-definite.

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Definition (Left Polar Decomposition)

The left polar decomposition of a matrix $A \in \mathbb{C}^{n \times m}$ $m \geq n$ has the form $A = HU$ where $H \in \mathbb{C}^{n \times n}$ is positive semi-definite and $U \in \mathbb{C}^{n \times m}$ has orthonormal columns.

Square Root of a Matrix

Theorem (The Square Root of a Matrix)

If A is a normal matrix then there exists a positive semi-definite matrix P such that $A = P^2$.

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Proof.

Suppose you have a normal matrix A of size n. Then A is orthonormally diagonalizable. This means that there is a unitary matrix S and a diagonal matrix B whose diagonal entries are the eigenvalues of A so that $A = SBS^*$ where $S^*S = I_n$. Since A is normal the diagonal entries of B are all positive, making B positive semi-definite as well. Because B is diagonal with real, non-negative entries we can easily define a matrix C so that the diagonal entries of C are the square roots of the eigenvalues of A. This gives us the matrix equality $C^2 = B$. Define P with the equality $P = SCS^*$.

The Theorem

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Theorem (Right Polar Decomposition)

For any matrix $A \in \mathbb{C}^{m\times n}$, where $m \geq n$, there is a matrix $U \in \mathbb{C}^{m\times n}$ with orthonormal columns and a positive semi-definite matrix $P \in \mathbb{C}^{n\times n}$ so that $A = UP$.

Example

A

$$A = \begin{bmatrix} 3 & 8 & 2 \\ 2 & 5 & 7 \\ 1 & 4 & 6 \end{bmatrix} \quad A^*A = \begin{bmatrix} 14 & 38 & 26 \\ 38 & 105 & 75 \\ 25 & 76 & 89 \end{bmatrix}$$

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S, S^{-1}, and C

$$S = \begin{bmatrix} 1 & 1 & 1 \\ -0.3868 & 2.3196 & 2.8017 \\ 0.0339 & -3.0376 & 2.4687 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 0.8690 & -0.3361 & 0.0294 \\ 0.0641 & 0.1486 & -0.1946 \\ 0.0669 & 0.1875 & 0.1652 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.4281 & 0 & 0 \\ 0 & 4.8132 & 0 \\ 0 & 0 & 13.5886 \end{bmatrix}$$

Example

P

$$P = \sqrt{A^*A} = S^*CS^{-1} = \begin{bmatrix} 1.5897 & 3.1191 & 1.3206 \\ 3.1191 & 8.8526 & 4.1114 \\ 1.3206 & 4.1114 & 8.3876 \end{bmatrix}$$

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U

$$U = \begin{bmatrix} 0.3019 & 0.9175 & -0.2588 \\ 0.6774 & -0.0154 & 0.7355 \\ -0.6708 & 0.3974 & 0.6262 \end{bmatrix}$$

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A

$$UP = \begin{bmatrix} 1.5897 & 3.1191 & 1.3206 \\ 3.1191 & 8.8526 & 4.1114 \\ 1.3206 & 4.1114 & 8.3876 \end{bmatrix} \begin{bmatrix} 0.3019 & 0.9175 & -0.2588 \\ 0.6774 & -0.0154 & 0.7355 \\ -0.6708 & 0.3974 & 0.6262 \end{bmatrix}$$

Polar Decomposition from SVD

Theorem (SVD to Polar Decomposition)

For any matrix $A \in \mathbb{C}^{m \times n}$, where $m \geq n$, there is a matrix $U \in \mathbb{C}^{m \times n}$ with orthonormal columns and a positive semi-definite matrix $P \in \mathbb{C}^{n \times n}$ so that $A = UP$.

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Proof.

$$
\begin{aligned}
A &= U_S S V^* \\
&= U_S I_n S V^* \\
&= U_S V^* V S V^* \\
&= UP
\end{aligned}
$$

□

Example Using SVD

Give Sage our A and ask to find the SVD

SVD

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Components

$$U_S = \begin{bmatrix} 0.5778 & 0.8142 & 0.0575 \\ 0.6337 & 0.4031 & 0.6602 \\ 0.5144 & 0.4179 & 0.7489 \end{bmatrix}$$

$$S = \begin{bmatrix} 13.5886 & 0 & 0 \\ 0 & 4.8132 & 0 \\ 0 & 0 & 0.4281 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.2587 & 0.2531 & 0.9322 \\ 0.7248 & 0.5871 & 0.3605 \\ 0.6386 & 0.7689 & 0.0316 \end{bmatrix}$$

Example Using SVD

U

$$U = U_S V^*$$

$$= \begin{bmatrix} 0.5778 & 0.8142 & 0.0575 \\ 0.6337 & 0.4031 & 0.6602 \\ 0.5144 & 0.4179 & 0.7489 \end{bmatrix} \begin{bmatrix} -0.2587 & -0.7248 & -0.6386 \\ 0.2531 & 0.5871 & -0.7689 \\ -0.9322 & 0.3605 & -0.0316 \end{bmatrix}$$

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P

$$P = VSV^*$$

$$= \begin{bmatrix} 0.2587 & 0.2531 & 0.9322 \\ 0.7248 & 0.5871 & 0.3605 \\ 0.6386 & 0.7689 & 0.0316 \end{bmatrix} \begin{bmatrix} 13.5886 & 0 & 0 \\ 0 & 4.8132 & 0 \\ 0 & 0 & 0.4281 \end{bmatrix} \begin{bmatrix} -0.2587 & -0.7248 & -0.6386 \\ 0.2531 & 0.5871 & -0.7689 \\ -0.9322 & 0.3605 & -0.0316 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5897 & 3.1191 & 1.3206 \\ 3.1191 & 8.8526 & 4.1114 \\ 1.3206 & 4.1114 & 8.3876 \end{bmatrix}$$

Geometry Concepts

Matrices

$A = UP$

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Matrices

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Complex Numbers

$z = re^{i\theta}$

Motivating Example

2×2

$$A = \begin{bmatrix} 1.300 & -.375 \\ .750 & .650 \end{bmatrix}$$

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Polar Decomposition

$$U = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.50 & 0.0 \\ 0.0 & 0.75 \end{bmatrix} = \sqrt{A^* A}$$

P and r

2×2

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

r

$$r = \sqrt{x^2 + y^2}$$

P and r

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r

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r Vector

$$\|\mathbf{r}\| = \sqrt{\mathbf{r}^*\mathbf{r}}$$

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P

$$P = \sqrt{A^*A}$$

iitit

Continuum Mechanics

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Computer Graphics

Iterative Methods for U

Newton Iteration

$$U_{k+1} = \frac{1}{2}(U_k + U_k^{-t}), \quad U_0 = A$$

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Frobenius Norm Accelerator

$$\gamma_{F_k} = \frac{\|U_k^{-1}\|_F^{\frac{1}{2}}}{\|U_k\|_F^{\frac{1}{2}}}$$

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Rotation Matrices

What's Up with U?

$$U = R_\theta R_\psi R_\kappa V^*$$

$$= \begin{bmatrix} \cos\psi\cos\kappa & \cos\psi\sin\kappa & -\sin\psi \\ \sin\theta\sin\psi\cos\kappa - \cos\theta\sin\kappa & \sin\theta\sin\psi\sin\kappa + \cos\theta\cos\kappa & \sin\theta\cos\psi \\ \cos\theta\sin\psi\cos\kappa + \sin\theta\sin\kappa & \cos\theta\sin\psi\sin\kappa - \sin\theta\cos\kappa & \cos\theta\cos\psi \end{bmatrix} V^*$$

P and r

r

$$r = \sqrt{x^2 + y^2}$$

P and r

r

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r Vector

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P and r

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$r = \sqrt{x^2 + y^2}$

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$\|\mathbf{r}\| = \sqrt{\mathbf{r}^* \mathbf{r}}$

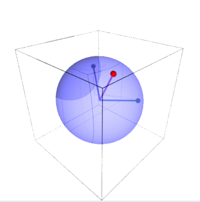
P

$P = \sqrt{A^* A}$

Ideal Example

U

$$\begin{bmatrix} .5 & .5 & -.7071 \\ -.1464 & .8536 & .5 \\ .8536 & -.1768 & .5 \end{bmatrix}$$



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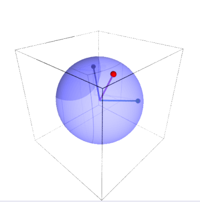
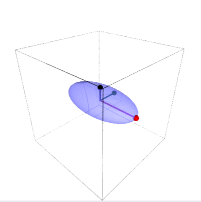


P

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .33 \end{bmatrix}$$



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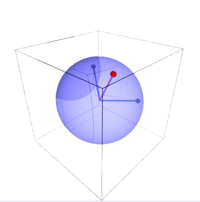

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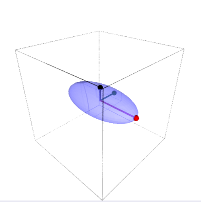
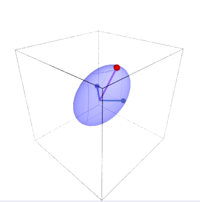


$A = UP$

$$\begin{bmatrix} .5 & .25 & -.2355 \\ -.1464 & .4268 & .1665 \\ .8536 & -.0884 & .1665 \end{bmatrix}$$



Applications

Use

Continuum Mechanics

Applications

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Another Use

Computer Graphics

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Conclusion

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