Solving Toeplitz Systems of Equations and the Importance of Conditioning

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Outline

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What is a Toeplitz Matrix

- A Toeplitz Matrix or Diagonal Constant Matrix is a *nxn* matrix where each of the descending diagonals are constant, where
- $$T_n = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-n+1} \\ t_1 & t_0 & \ddots & t_{-2} \\ \vdots & \ddots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & \cdots & t_0 \end{bmatrix}$$
- eigenvectors of Toeplitz matrices are sines and cosines
- Toeplitz matrices are also related to Fast Fourier Tranforms (FFT) and when looking at images and signals processing, Fourier Transforms, Hilbert Spaces, and problems involving trigometric moments.

What is a Toeplitz Matrix

Definition 1.1

Let A be an n x n matrix such that A is persymmetric if it is symmetric about its anti-diagonal

Definition 1.2

Let A be a n x n matrix such that $A$is centrosymmetric if it is symmetric about the center

Definition 1.3

Let A be a n x n matrix. A is bisymmetric if only if A is centrosymmetric and either symmetric or antisymmetric

Outline

Conditioning of a Matrix

What is conditioning? Why does it matter?

The Conditioning Number of a Matrix

$$\kappa(A) = \|A\| \, \|A^{-1}\| \geq 1 \tag{1}$$

- if $\kappa(A)$ is large than the matrix A is ill-conditioned
- if $\kappa(A)$ is small than the matrix A is well-conditioned

Outline

Matrix Norms How do we calculate a Matrix Norm? There are three commonly used norms

1-Norm

Let A be an m x n matrix. The 1-norm, $\left\|A\right\|_1$ is equal to the maximum column sum or for $1 \leq j \leq n$ and a_j is the jth column of A

$$\left\|A\right\|_1 = \max_j \sum_{k=1}^{n} a_{kj} \tag{2}$$

Matrix Norms

2-Norm

Let A be an m x n matrix. The 1-norm, $||A||_2$ is equal to the largest singular value of A

$$||A||_2 = \max_i \delta \quad (3)$$

Matrix Norms

∞-Norm

Let A be an m x n matrix. The 1-norm, $||A||_\infty$ is equal to the maximum row sum or for $1 \leq i \leq m$ and a_i is the ith row of A

$$||A||_\infty = \max_i \sum_{k=1}^{m} a_{ik} \tag{4}$$

Outline

Block Gaussian Elimination

Why would we choose block Gaussian elimination compared to other algorithms? What is block Gaussian elimination?

Block Gaussian Elimination

Suppose we have the system $Tx = b$ where T is Toeplitz, symmetric and nonsingular. Then partition T

$$Tx = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x} \\ \check{x} \end{bmatrix} = \begin{bmatrix} \hat{b} \\ \check{b} \end{bmatrix} = b \tag{5}$$

where x and b are nx1, A is $(k$x$k)$, B is kx$(n-k)$, C is $(n-k)$xk, D is $(n-k)$x$(n-k)$, $\hat{x}$ and $\hat{b}$ are kx1 and $\check{x}$ and $\check{b}$ are $(n-k)$x1.

Block Gaussian Elimination

we then use block Gaussian elimination to break our new partition matrix into an upper and lower triangular matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & \Delta \end{bmatrix} \tag{6}$$

Where $\Delta = D - CA^{-1}B$, and

$$\begin{bmatrix} A & B \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \check{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{b} \\ \check{b} \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \check{x} - CA^{-1}\hat{x} \end{bmatrix} \tag{7}$$

Block Gaussian Elimination

We then solve for $\hat{x}$ and $\breve{x}$ by

1. Solving $AX = C$ for X, where X is $(n-k)xk$ matrix
2. Forming $\Delta = D - XB$
3. Forming $\breve{c} = \breve{b} - X\hat{b}$
4. Solving $\Delta\breve{x} = \breve{c}$ for $\breve{x}$
5. Forming $\hat{c} = \hat{b} - B\hat{x}$ and
6. Solving $A\hat{x} = \hat{c}$ for $\hat{x}$.

Though this method is pretty stable there can be problems

Block Gaussian Elimination

The biggest problem with block Gaussian elimination is that even if T is well-conditioned, A can be ill-conditioned. There is only one class of matrices that proves that to be true- symmetric, positive-definite matrices, or Hermitian in the complex case.

Block Gaussian Elimination

let us take the 2-norm of both T and A

$$\kappa_2(T) = \frac{\sigma_{max}(T)}{\sigma_{min}(T)} \tag{8}$$

$$\kappa_2(A) = \frac{\sigma_{max}(A)}{\sigma_{min}(A)} \tag{9}$$

where σ_{max} is the largest singular value and σ_{min} is the smallest. Since T and A are symmetric positive definite, $\sigma_{max}(T) = \lambda_{max}(T)$, $\sigma_{min}(T) = \lambda_{min}(T)$, $\sigma_{max}(A) = \lambda_{max}(A)$, $\sigma_{min}(A) = \lambda_{min}(A)$, where λ_{max} is the largest eigenvalue and λ_{min} is the smallest

Block Gaussian Elimation

Cauchy Interlace Theorem

Let A be a symmetric nxn matrix. Let B an mxm matrix where $m \leq n$. Let B also be the compression of A. If the eigenvalues of A are $\alpha_1 \leq \cdots \leq \alpha_n$, and those of B are $\beta_1 \leq \cdots \leq \beta_j \leq \cdots \leq \beta_m$ then for all $j < m + 1$

Block Gaussian Elimination

From the Cauchy Interlace Theorem we know,

$$0 < \lambda_{min}(T) \leq \lambda_{min}(A) \leq \lambda_{max}(A) \leq \lambda_{max}(T) \tag{10}$$

Thus,

$$\kappa_2(A) = \frac{\lambda max(A)}{\lambda min(A)} \leq \frac{\lambda max(T)}{\lambda min(T)} = \kappa_2(T) \tag{11}$$

Therefore if T is well-conditioned then A is also well-conditioned.

Outline

Large Example Time

More fun than Disneyland

Consider the matrix

$$T = \begin{bmatrix} 1 & 2 & 0 & -1 & 5 & 8 \\ 2 & 1 & 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 1 & 2 & 0 \\ 5 & -1 & 0 & 2 & 1 & 2 \\ 8 & 5 & -1 & 0 & 2 & 1 \end{bmatrix} \quad (12)$$

where T is symmetric, nonsingular and positive-definite.

Large Example Time

Before partitioning the matrix, check the conditioning

$$\|T\|_1 = 15 \tag{13}$$

$$\|T\|_2 \approx 12.822 \tag{14}$$

$$\|T\|_\infty = 15 \tag{15}$$

$$\|T^{-1}\|_1 \approx .284 \tag{16}$$

$$\|T^{-1}\|_2 \approx .784 \tag{17}$$

$$\|T^{-1}\|_\infty \approx .284 \tag{18}$$

Large Example

Knowing all three matrix norms, we compute the conditioning numbers

$$\kappa(T)_1 = \left\| T \right\|_1 \left\| T^{-1} \right\|_1 = (15)(.284) = 4.26 \tag{19}$$

$$\kappa(T)_2 = \left\| T \right\|_2 \left\| T^{-1} \right\|_2 = (12.822)(.784) = 10.05 \tag{20}$$

$$\kappa(T)_\infty = \left\| T \right\|_\infty \left\| T^{-1} \right\|_\infty = (15)(.284) = 4.26 \tag{21}$$

Since $\kappa(T)$ is relatively small then T is well-conditioned.

Large Example

Paritition T

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 5 & 8 \\ 0 & -1 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 2 \\ 5 & -1 & 0 \\ 8 & 5 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Large Example

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \check{x} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \check{b} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

Large Example

now we calculate CA^{-1} and Δ

$$CA^{-1} = \begin{bmatrix} -\frac{11}{7} & \frac{2}{7} & \frac{10}{7} \\ \frac{13}{7} & \frac{11}{7} & -\frac{22}{7} \\ \frac{38}{7} & \frac{9}{7} & -\frac{25}{7} \end{bmatrix} \tag{22}$$

$$\Delta = D - CA^{-1}B = \begin{bmatrix} -\frac{24}{7} & \frac{71}{7} & \frac{88}{7} \\ \frac{71}{7} & -\frac{47}{7} & -\frac{167}{7} \\ \frac{88}{7} & -\frac{167}{7} & -\frac{367}{7} \end{bmatrix} \tag{23}$$

Large Example

Now we solve for x

$$\breve{c} = \breve{b} - CA^{-1}\hat{b} = \begin{bmatrix} \frac{19}{7} \\ -\frac{67}{7} \\ -\frac{65}{7} \end{bmatrix} \tag{24}$$

$$\breve{x} = \Delta^{-1}\breve{c} = \begin{bmatrix} -\frac{9418}{7807} \\ -\frac{21}{7807} \\ -\frac{866}{7807} \end{bmatrix} \approx \begin{bmatrix} -1.2063532727 \\ -0.00268989368515 \\ -0.110926091969 \end{bmatrix} \tag{25}$$

Large Example

Since we have $\check{x}$, we can finally solve for $\hat{x}$

$$\hat{x} = A^{-1}(\hat{b} - B\check{x}) = \begin{bmatrix} -\frac{22}{7807} \\ \frac{2722}{7807} \\ \frac{4719}{7807} \end{bmatrix} \approx \begin{bmatrix} -0.00281798386064 \\ 0.348661457666 \\ 0.604457538107 \end{bmatrix} \tag{26}$$

Large Example

$$x = \begin{bmatrix} \hat{x} \\ \check{x} \end{bmatrix} = \begin{bmatrix} -0.00281798386064 \\ 0.348661457666 \\ 0.604457538107 \\ 1.2063532727 \\ -0.00268989368515 \\ -0.110926091969 \end{bmatrix} \tag{27}$$

Outline

Final Thoughts

- Block Gaussian Elimination uses $O(n^2)$ flops while preserving Toeplitz structure
- the block matrix A must be proven to be well-conditioned or else it can ruin your solution(s)

Resources

If you want to improve this style

- Trefethen, L. N., Bau, D. (1997). Numerical linear algebra. Philadelphia, PA: Society for Industrial and Applied Mathematics.
- Horn, R., Johnson, C. (1985). Matrix analysis. (1 ed.). New York, New York: Press Syndicate of the University of Cambridge.
- Bunch, J. (1985). Stability of Methods for Solving Toeplitz Systems of Equations. Society for Industrial and Applied Mathematics, 6(2), 349-364.
- Luk, F., Qiao, S. (1996). A Symmetric Rank-Revealing Toeplitz Matrix Decomposition. Journal of VLSI Signal Processing, (8), 1-9.