The Pseudoinverse

Moore-Penrose Inverse and Least Squares

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Outline

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What is the Generalized Inverse?

- Any inverse-like matrix

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Example

$ALA = A(LA) = AI = A$
$ARA = (AR)A = IA = A$

Defining the Pseudoinverse

Definition

If $A \in \mathbb{M}^{n \times m}$, then there exists a unique $A^+ \in \mathbb{M}^{m \times n}$ that satisfies the four Penrose conditions:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $A^+A = (A^+A)^*$ Hermitian
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Properties of the Pseudoinverse

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- $(AB)^{+} = B^{+}A^{+}$

Properties of the Pseudoinverse

- For any $A \in \mathbb{C}^{n \times m}$ there exists a $A^{+} \in \mathbb{C}^{m \times n}$
- $N(A^{*}) = N(A^{+})$ and $R(A^{*}) = R(A^{+})$

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- $R(A) \oplus N(A^{+}) = \mathbb{C}^{n}$ and $R(A^{+}) \oplus N(A) = \mathbb{C}^{m}$

Properties of the Pseudoinverse

- For any $A \in \mathbb{C}^{n \times m}$ there exists a $A^+ \in \mathbb{C}^{m \times n}$
- $N(A^*) = N(A^+)$ and $R(A^*) = R(A^+)$
- $R(A) \oplus N(A^+) = \mathbb{C}^n$ and $R(A^+) \oplus N(A) = \mathbb{C}^m$
- HH^+ is an orthogonal projection onto $R(A)$, and using similar H^+H is an orthogonal projection onto $N(A)$.

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QR Decomposition

There are two ways of constructing the Pseudoinverse using QR.

- If A is $n \times m$ and $n > m$ with rank equal to m then $A = Q \begin{bmatrix} R_1 \\ \mathcal{O} \end{bmatrix}$

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- This only works if rank is equal to the minimum of m and$n$

QR Decomposition

The second way finds the pseudoinverse for the singular case

- If A is $n \times n$ and rank is less than n then $A = Q \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$

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- If A is $n \times n$ and rank is less than n then $A = Q \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$
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QR Example

Example

- Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

QR Example

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- Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$
- $Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}$ $R = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

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- $Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}$ $R = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$
- $R_1^{-1} = \begin{bmatrix} 1/2 & -3/10 & -2/5 \\ 0 & 1/5 & 1/10 \\ 0 & 0 & 1/4 \end{bmatrix}$

QR Example

Example

- $A^{+} = \begin{bmatrix} R_1^{-1} & \mathcal{O}^* \end{bmatrix} Q^*$

QR Example

Example

- $A^{+} = \begin{bmatrix} R_1^{-1} & \mathcal{O}^* \end{bmatrix} Q^*$
- $A^{+} = \begin{bmatrix} 1/5 & 3/10 & -1/10 & 3/5 \\ -1/20 & 1/20 & 3/20 & -3/20 \\ 1/8 & -1/8 & 1/8 & -1/8 \end{bmatrix}$
- $A^{+}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

SVD Construction

- SVD can be represented by $A = UDV^*$
- or by $A = \sum_{i=1}^{r} s_i x_i {y_i}^*$

SVD Construction

- SVD can be represented by $A = UDV^*$
- or by $A = \sum_{i=1}^{r} s_i x_i {y_i}^*$
- the pseudoinverse can be found by $A^+ = VD^+U^*$
- or $A^+ = \sum_{i=1}^{r} s_i^{-1} y_i {x_i}^*$

SVD Example

Example

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{3} & 0 \end{bmatrix}$ then $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

SVD Example

Example

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{3} & 0 \end{bmatrix}$ then $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- Solving for D^+, $D^+ = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and

  $A^+ = \begin{bmatrix} .25 & 00.433012701892 & \\ 0 & 1 & 0 \end{bmatrix}$
- $A^+A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Least Squares Review

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- $x_0 = A^{+}b = (A^{*}A)^{-1}A^{*}b = A^{+}b$
- A has full column rank, $n > m$. A^{+} solves the least squares solution
- similarly $A^{+} = A^{*}(AA^{*})^{-1}$ when A has full row rank.

Digital Image Restoration

- Recovery of a degraded images

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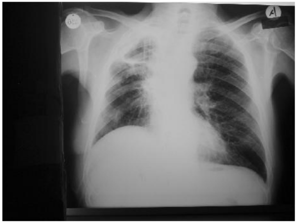
- Recovery of a degraded images
- Many algorithms are used for a variety of reasons
- The Improvement in signal-to-noise ratio (ISNR) and required computational time compare algorithms
- Moore-Penrose inverse is one of the most efficient algorithms

Digital Image Restoration Example

- $x_{in} = H^{+} x_{out}$
- Where H is the matrix representation of how the image was degraded by a uniform linear motion

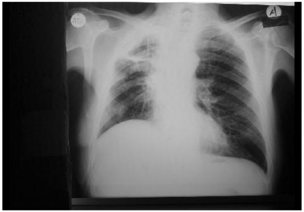
Digital Image Restoration



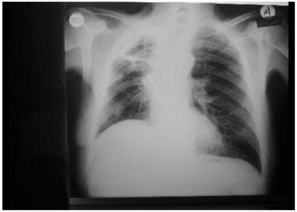
Original image

Degraded image

Digital Image Restoration



Generalized inverse reconstructed image



Lagrange reconstructed image

Digital Image Restoration

Table 1: ISNR and computational time results for 10 random matrices.

a	Ginv ISNR	Lagrange ISNR	Ginv computation time	Lagrange computation time
5	0.3534	0.3587	6.4210	9.2040
10	0.3485	0.3635	7.0780	10.0970
15	0.3484	0.3618	8.7940	10.6780
20	0.3475	0.3568	9.1990	11.6580
25	0.3457	0.3698	9.7760	12.0000
30	0.3537	0.3643	10.1810	12.5540
35	0.3546	0.3651	10.7710	12.5320
40	0.3524	0.3623	11.1230	13.1120
45	0.3642	0.3660	11.8990	14.3230
50	0.3559	0.3778	12.1400	14.9670

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