Multilinear Algebra

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Overview

1. Basics
   - Multilinearity
   - Dual Space
2. Tensors
   - Tensor Product
   - Basis of $\mathcal{T}^p_q(V)$
3. Component Representation
   - Kronecker Product
   - Components
   - Comparison

Multilinear Functions

Definition

A function $f : V \mapsto W$, where V and W are vector spaces over a field F, is linear if for all x, y in V and all α, β in F

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

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A function $f : V \times U \mapsto W$, where V, U, and W are vector spaces over a field F, is bilinear if for all x, y in V and all α, β in F

$$f(\alpha x_1 + \beta x_2, y) = \alpha f(x_1, y) + \beta f(x_2, y), \text{ and}$$
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A function $f : V_1 \times \cdots \times V_s \mapsto W$, where $\{V_i\}_{i=1}^{s}$ and W are vector spaces over a field F, is *s-linear* if for all x_i, y_i in V_i and all α, β in F

$$f(v_1, \ldots, \alpha x_i + \beta y_i, \ldots, v_s) =$$
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for all indices i in $\{1, \ldots, s\}$.

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For any collection of vector spaces $\{V_i\}_{i=1}^{s}$, and any collection of linear functions $f_i : V_i \mapsto \mathbb{R}$, the function

$$f(v_1, \ldots v_s) = \prod_{i=1}^{s} (f_i(v_i))$$

is s-linear.

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- $(V^*)^*$ is identical to V.
- Notation: $\langle v^*, v \rangle$ denotes the value of v^* evaluated at v. For our purposes, consider it the inner product of v and $(v^*)^T$.

Tensors

Definition

A tensor of order (p, q) is a $(p + q)$-linear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{p \text{ times}} \times \underbrace{V \times \cdots \times V}_{q \text{ times}} \mapsto \mathbb{R}.$$

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- $\mathcal{T}^0_1(V) = V^*$, $\mathcal{T}^1_0(V) = V$, and $\mathcal{T}^1_1(V) \cong L(V : V)$.
- That is, lower order tensors are the 1 and 2 dimensional arrays we usually work with.

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- Recall that this is a special case of our earlier example, as $(v \otimes v^*)$ is the product of two linear functions.

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- However, we can use simple tensors to build a basis of $\mathcal{T}_q^p(V)$.

Theorem

For any basis of V, $B = \{e_i\}_{i=1}^N$, there exists a unique dual basis of V^* relative to B, denoted $\{e^j\}_{j=1}^N$ and defined as

$$\langle e^j, e_i \rangle = \delta_i^j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

Theorem

For any basis $\{e_i\}_{i=1}^N$ of V, and the corresponding dual basis $\{e^j\}_{j=1}^N$ of V^*, the set of simple tensors

$$\{e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q}\}$$

for all combinations of $\{i_k\}_{k=1}^p \in \{1, \ldots, N\}$ and $\{j_z\}_{z=1}^q \in \{1, \ldots, N\}$, forms a basis of $\mathcal{T}^p_q(V)$.

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- The size of this basis is $N^{(p+q)}$.
- Simplified proof in my paper, but our relation of linear dependence is nasty ($p + q$ nested sums).

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For two matrices $A_{m\times n}$ and $B_{p\times q}$, the Kronecker product of A and B is defined as

$$A \otimes B = \begin{pmatrix} [A]_{1,1}B & [A]_{1,2}B & \cdots & [A]_{1,n}B \\ [A]_{2,1}B & [A]_{2,2}B & \cdots & [A]_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ [A]_{m,1}B & [A]_{m,2}B & \cdots & [A]_{m,n}B \end{pmatrix}$$

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- Can be represented by 2-dimensional array, but we consider this product to be a list of lists, table of lists, list of tables, table of tables, ect.

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In general, we define the components of $T \in \mathcal{T}^p_q(V)$ to be the $(p+q)$-indexed scalars

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- For vectors, this is exactly how we define components $(\langle v, e_i\rangle = [v]_i)$.

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In general, we define the components of $T \in \mathcal{T}_q^p(V)$ to be the $(p+q)$-indexed scalars

$$A^{i_1,\ldots,i_p}_{j_1,\ldots,j_q} = A(e^{i_1}, \ldots, e^{i_p}, e_{j_1}, \ldots, e_{j_q}).$$

- For vectors, this is exactly how we define components ($\langle v, e_i \rangle = [v]_i$).
- If T is a simple tensor, then the $(p+q)$-dimensional array formed by $A^{i_1,\ldots,i_p}_{j_1,\ldots,j_q}$ is equal to the Kronecker product of the vectors and covectors which make up T.

Example

For $V = \mathbb{R}^2$, consider the vectors $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v^* = \begin{bmatrix} 2 & 1 \end{bmatrix}$, and $w^* = \begin{bmatrix} 1 & 3 \end{bmatrix}$. Let $A = u \otimes v^* \otimes w^*$ and consider

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$$\begin{aligned}
A_1^{1,1} &= A(e^1, e_1, e_1) \\
&= \langle \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle \langle \begin{bmatrix} 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \langle \begin{bmatrix} 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \\
&= 2 \\
A_1^{2,1} &= A(e^1, e_2, e_1) = 1 \\
A_1^{1,2} &= A(e^1, e_1, e_2) = 6 \\
&\vdots
\end{aligned}$$

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$$\begin{aligned} u \otimes v^* \otimes w^* &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2\begin{bmatrix} 1 & 3 \end{bmatrix} & 1\begin{bmatrix} 1 & 3 \end{bmatrix} \\ 2\begin{bmatrix} 1 & 3 \end{bmatrix} & 1\begin{bmatrix} 1 & 3 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 2 & 6 \\ 2 & 6 \end{bmatrix} & \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \end{bmatrix}, \end{aligned}$$

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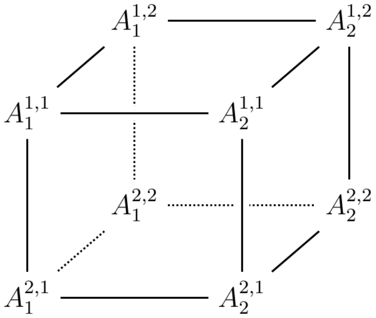

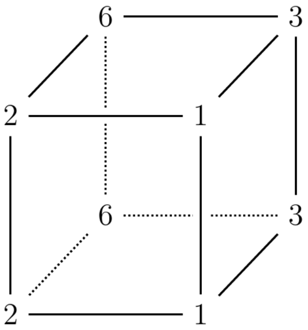

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The End