Rational Canonical Form

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Why do we need Rational Canonical Form?

Consider the matrix over $\mathbb{R}$,

$$A = \begin{pmatrix} 5 & 6 & 3 & 4 \\ -1 & 9 & 2 & 7 \\ 4 & -2 & -8 & 10 \\ 21 & -14 & 6 & 3 \end{pmatrix}$$

- This matrix has characteristic polynomial $x^4 + 9x^3 - 97x^2 + 567x - 9226$

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- This matrix has characteristic polynomial $x^4 + 9x^3 - 97x^2 + 567x - 9226$
- Can not find Jordan Canonical Form for this matrix.

What is Rational Canonical Form?

Recall that a companion matrix for a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_0$ is the matrix of the form:

$$\begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & ... & 0 & -a_1 \\ 0 & 1 & ... & 0 & -a_2 \\ 0 & 0 & ... & 0 & -a_3 \\ . & . & ... & . & . \\ . & . & ... & . & . \\ 0 & 0 & ... & 1 & -a_{n-1} \end{pmatrix}$$

A matrix in Rational Canonical Form is a matrix of the form

$$\begin{pmatrix} C[f_n] & & & \\ & C[f_{n-1}] & & \\ & & \ddots & \\ & & & C[f_1] \end{pmatrix}$$

Where $C[f_i]$ is a companion matrix for the polynomial f_i.
Furthermore, $f_n | f_{n-1} | ... | f_1$.

k[x]-modules

Definition

Recall that a k[x]-module is a module with scalars from the ring k[x] and scalar multiplication defined as follows:

Given $f(x) \in k[x]$, $f(x)v = \sum_{i=0}^{n} a_i x^i v = \sum_{i=0}^{n} a_i T^i(v) = f(T)(v)$.

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- We can think of this as the module associated with the linear transformation T

Definition

Given an R-module, M, and $m \in M$, the **annihilator** of $m \in M$ is:

$$\text{ann}(m) = \{r \in R : rm = 0\}.$$

Theorem

Given a vector space V over a field F and a linear transformation $T : V \to V$, the $F[x]$-module, V^T, is a torsion module.

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Proof.

the set $\{v, T(v), T^2(v), ..., T^n(v)\}$ is linearly dependent since it contains $n + 1$ vectors.

$$g(x) = \sum_{i=0}^{n} a_i x^i \in \text{ann}(v)$$ □

Definition

If M is an R-module, then a **submodule** N of M, denoted $N \subseteq M$ is an additive subgroup N of M closed under scalar multiplication. That is, $rn \in N$ for $n \in N$ and $r \in R$.

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Theorem

Given a vector space V over a field F and a linear transformation, $T : V \to V$, a submodule W of the $F[x]$-module V^T is a T-invariant subspace. More specifically, $T(W) \subseteq W$.

The Minimal Polynomial and k[x]-modules

Definition

The **annihilator** of a module, M, is:

$$\text{ann}(M) = \{r \in R : rm = 0 \text{ for all } m \in M\}$$

Definition

The **Minimal Polynomial** of a matrix A, denoted $m_A(x)$, is the unique monic polynomial of least degree such that $m_A(A) = 0$.

The Minimal Polynomial and k[x]-modules

- These two terms are related for k[x]-modules

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$$\begin{aligned} \text{ann}(V^T) =& \{f(x) \in F[x] | f(x)v = 0 \text{ for all } v \in V\} \\ =& \{f(x) \in F[x] | f(T)v = 0 \text{ for all } v \in V\} \\ =& \{f(x) \in F[x] | f(T) = 0\} \end{aligned}$$

- We can use these terms synonymously

Matrix Representation of Cyclic Submodules

Definition

Given an R-module, M, and an element $m \in M$, the **cyclic submodule** generated by m is

$$\langle m \rangle = \{rm : r \in R\}$$

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Matrix Representation of Cyclic Submodules

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Given an R-module, M, and an element $m \in M$, the **cyclic submodule** generated by m is

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- Since a submodule, W, of a k[x]-module is T-invariant, we can examine the matrix representation $T|_W$
- Let us look at T restricted to cyclic submodules of $k[x]$-modules

Theorem

Let $W = \langle w \rangle$ be a cyclic submodule of the $F[x]$-module V^T and $deg(m_T|W(x)) = n$. Then the set $\{T^{n-1}(w), T^{n-2}(w), ..., T(w), w\}$ is a basis for W.

Proof.

- By the division algorithm, we can write any polynomial $f(x) = m(x)q(x) + r(x)$ where $m(x)$ is the minimal polynomial of $T|_W$ with deg=n and deg$(r(x)) < n$
- so, for any $w_1 \in W$,

$$\begin{aligned} w_1 =& r(x)w \\ =& r(T)w \\ =& a_{n-1}T^{n-1}(w) + a_{n-2}T^{n-2}(w) + ... + a_0(w). \end{aligned}$$

□

Proof cont.

- Consider the relation of linear dependence: $a_{n-1}T^{n-1}(w) + a_{n-2}T^{n-2}(w) + ... + a_0(w) = 0$
- $a_{n-1}T^{n-1}(w) + a_{n-2}T^{n-2}(w) + ... + a_0(w) = p(x)w$, $deg(p(x)) < deg(m(x))$

- Now consider the matrix representation of $T|_W$ relative to the basis $\{w, T(w), ..., T^{n-1}(w)\}$

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$$\begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & ... & 0 & -a_1 \\ 0 & 1 & ... & 0 & -a_2 \\ 0 & 0 & ... & 0 & -a_3 \\ . & . & ... & . & . \\ . & . & ... & . & . \\ 0 & 0 & ... & 1 & -a_{n-1} \end{pmatrix}$$

Primary Decomposition

Theorem

Let M be a finitely generated torsion module over a principal ideal domain, D, and let $ann(M) = \langle u \rangle, u = p_1^{e_1} p_2^{e_2} ... p_n^{e_n}$ where each p_i is prime in D. Then

$$M = M_{p_1} \oplus M_{p_2} \oplus ... \oplus M_{p_n}$$

where $M_{p_i} = \{v \in V : p_i^e v = 0\}$.

Cyclic Decomposition

Theorem

Let M be a primary, finitely generated torsion module over a principle ideal domain, R with $ann(M) = \langle p^e \rangle$*, then M is the direct sum,*

$$M = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \ldots \oplus \langle v_n \rangle$$

where $ann(\langle v_i \rangle) = p^{e_i}$ *and the terms in each cyclic decomposition can be arranged such that*

$$ann(v_1) \supseteq ann(v_2) \supseteq \ldots \supseteq ann(v_n).$$

Therefore, we can write:
$V^T = M_{p_1} \oplus M_{p_2} \oplus \ldots M_{p_n} =$
$(\langle v_{1,1} \rangle \oplus \langle v_{1,2} \rangle \oplus \ldots \oplus \langle v_{1,k_1} \rangle) \oplus \ldots \oplus (\langle v_{n,1} \rangle \oplus \ldots \oplus \langle v_{n,k_n} \rangle)$

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- $\text{ann}(\langle v_{i,j} \rangle) = p_i^{e_{i,j}}$

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- $\text{ann}(\langle v_{i,j} \rangle) = p_i^{e_{i,j}}$
- $p_i^{e_i} = p_i^{e_{i,1}} \geq p_i^{e_{i,2}} \geq \ldots \geq p_i^{e_{i,k_i}}$

The Invariant Factor Decomposition

- We can rearange these cyclic subspaces into the following groups

$$W_1 = \langle v_{1,1} \rangle \oplus \langle v_{2,1} \rangle \oplus \ldots \oplus \langle v_{n,1} \rangle$$
$$W_2 = \langle v_{1,2} \rangle \oplus \langle v_{2,2} \rangle \oplus \ldots \oplus \langle v_{n,2} \rangle$$
$$\vdots$$

- each W_i is cyclic with order $p^{e_{1,i}} p^{e_{2,i}} \ldots p^{e_{j,i}} = d_i$
- Each d_i is called an **invariant factor** of V^T
- Notice that since $d_1 = p_1^{e_{1,1}} p_2^{e_{2,1}} \ldots p_n^{e_{n,1}}, d_2 = p_1^{e_{1,2}} p_2^{e_{2,2}} \ldots p_n^{e_{n,2}}, \ldots$ We can conclude that $d_n | d_{n-1} | \ldots | d_1$

Example

Suppose that W is a torsion module with order $p_1^{e_1} p_2^{e_2} p_3^{e_3}$

- $W = M_{p_1} \oplus M_{p_2} \oplus M_{p_3}$
- Suppose that $M_{p_1} \oplus M_{p_2} \oplus M_{p_3} = (\langle v_{1,1} \rangle \oplus \langle v_{1,2} \rangle \oplus \langle v_{1,3} \rangle) \oplus (\langle v_{2,1} \rangle \oplus \langle v_{2,2} \rangle) \oplus (\langle v_{3,1} \rangle)$
- Then the $\langle p_1^{e_1} \rangle = ann(v_{1,1}) \supseteq ann(v_{1,2}) \supseteq ann(v_{1,3}), \langle p_2^{e_2} \rangle = ann(v_{2,1}) \supseteq ann(v_{2,2}), p_3^{e_3} = ann(v_{3,1})$.
- $W = (\langle v_{1,1} \rangle \oplus \langle v_{2,1} \rangle \oplus \langle v_{3,1} \rangle) \oplus (\langle v_{1,2} \rangle \oplus \langle v_{2,2} \rangle) \oplus (\langle v_{1,3} \rangle)$
- $d_1 = p_1^{e_{1,1}} p_2^{e_{2,1}} p_3^{e_{3,1}} = p_1^{e_1} p_2^{e_2} p_3^{e_3}$, $d_2 = p_1^{e_{1,2}} p_2^{e_{2,2}}$, and $d_3 = p_1^{e_{1,3}}$

Rational Canonical Form

- Given any matrix, we can realize this matrix as the linear transformation, T, associated with the $k[x] - module$, V^T

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- Given any matrix, we can realize this matrix as the linear transformation, T, associated with the $k[x] - module$, V^T
- The first invariant factor will be the minimum polynomial
- Each invariant factor will be a factor of the minimum polynomial

Example

Consider the matrix,

$$\begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{pmatrix}$$

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Consider the matrix,

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- characteristic polynomial is $x^3 + 6x^2 + 12x + 8 = (x+2)^3$
- minimal polynomial is $(x+2)^2$ since $(A+2I)^2 = 0$
- invariant factors are $(x+2)^2$ and $x+2$

Example cont.

Therefore, the rational canonical form of this matrix is:

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & 1 & -4 \end{pmatrix}$$