Math 290
Exam 3
Chapter M

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to manipulate matrices. More explicit instructions are given on the two questions (only) where you may use Sage.

1. Rewrite the system of equations below as a vector equality and using a matrix product (illustrating Theorem SLEMM). Then use the inverse of a matrix to find the solution set of this version of the system. You may only use the Sage command to row-reduce matrices (`.rref()`) as justification for your work. There will be no credit for solutions obtained by other methods. (15 points)

$$\begin{aligned} -2x_1 - 5x_2 - 6x_3 &= 1 \\ 3x_1 + 7x_2 + 8x_3 &= 1 \\ 2x_1 + 5x_2 + 7x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} -2 & -5 & -6 \\ 3 & 7 & 8 \\ 2 & 5 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad A\underset{\sim}{x} = \underset{\sim}{b}$$

$$A^{-1}A\underset{\sim}{x} = A^{-1}\underset{\sim}{b} \rightarrow I\underset{\sim}{x} = A^{-1}\underset{\sim}{b} \rightarrow \underset{\sim}{x} = A^{-1}\underset{\sim}{b}$$

$$[A \mid I_3] \xrightarrow{\text{RREF}} \left[I_3 \,\middle|\, \begin{matrix} 9 & 5 & 2 \\ -5 & -2 & -2 \\ 1 & 0 & 1 \end{matrix} \right]$$ So by CINM & OSIS, $A^{-1} = \begin{bmatrix} 9 & 5 & 2 \\ -5 & -2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$

then $A^{-1}\underset{\sim}{b} = 1\begin{bmatrix} 9 \\ -5 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ -2 \end{bmatrix}$

2. (**Work the page 2 problem next**, then return to this problem. It is simply here for space reasons.)
Give a careful proof that matrix addition is commutative using the definition of matrix equality (Definition ME). That is: if A and B are $m \times n$ matrices, prove that $A + B = B + A$. Give a reason for each step of your proof. (15 points)

For $1 \le i \le m$, $1 \le j \le n$,

$$\begin{aligned} [A+B]_{ij} &= [A]_{ij} + [B]_{ij} && \text{Defn MA} \\ &= [B]_{ij} + [A]_{ij} && \text{Property CACN} \\ &= [B+A]_{ij} && \text{Defn MA} \end{aligned}$$

3. Consider the matrix A below and questions about its column space, $\mathcal{C}(A)$, row space, $\mathcal{R}(A)$, and null space, $\mathcal{N}(A)$. You may use results from Sage's reduced row echelon form and extended echelon form commands (be sure to give the output from Sage as part of your answer) as justification. (50 points)

$$A = \begin{bmatrix} 3 & -3 & 3 & -9 \\ -1 & 2 & 0 & 6 \\ -1 & 1 & -1 & 3 \end{bmatrix}$$

(a) Give a set T, so that $\langle T\rangle = \mathcal{C}(A)$, and which illustrates the definition of a column space.

$$T = \left\{ \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -9 \\ 6 \\ 3 \end{bmatrix} \right\}$$

(b) Find a set R so that (a) R is linearly independent, (b) $\langle R\rangle = \mathcal{C}(A)$, and (c) each vector of R is a column of A.

$$A \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \left\{ \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$D = \{1, 2\}$

has the desired properties by Theorem BCS.

(c) Find a set S so that (a) S is linearly independent, (b) $\langle S\rangle = \mathcal{C}(A)$, and (c) the computation begins with the transpose of A.

$$A^t \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{C}(A) = \mathcal{R}(A^t) = \left\langle \underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ -1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}}_{S} \right\rangle$$

By Theorem BRS

(d) Find a set U so that (a) U is linearly independent, (b) $\langle U\rangle = \mathcal{C}(A)$, and (c) the computation uses the L matrix of the extended echelon form of A.

$$[A|I_3] \xrightarrow{RREF} \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right] \Rightarrow L = [1\ 0\ 3]$$

$$\mathcal{C}(A) = \mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \underbrace{\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}}_{U} \right\rangle$$

(e) Find a set V so that (a) V is linearly independent, (b) $\langle V\rangle = \mathcal{R}(A)$.

non zero rows from RREF in (b), as columns, Theorem BRS

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

(f) Find a set W so that (a) W is linearly independent, (b) $\langle W\rangle = \mathcal{N}(A)$.

Theorem BNS from last Chapter, use rref from (b)

$$W = \left\{ \begin{bmatrix} \\ \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are both solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$. Prove that $\mathbf{x} - \mathbf{y}$ is in the null space of A, that is $\mathbf{x} - \mathbf{y} \in \mathcal{N}(A)$. (10 points)

By theorem SLEMM, we know $A\underset{\sim}{x} = \underset{\sim}{b}$ & $A\underset{\sim}{y} = \underset{\sim}{b}$.

Then $A(\underset{\sim}{x} - \underset{\sim}{y}) = A\underset{\sim}{x} - A\underset{\sim}{y}$ Theorem MMDAA

$= \underset{\sim}{b} - \underset{\sim}{b}$ Hypothesis

$= \underset{\sim}{0}$

So $\underset{\sim}{x} - \underset{\sim}{y}$ is a solution to $\mathcal{LS}(A, \underset{\sim}{0})$ by theorem SLEMM. Thus, $\underset{\sim}{x} - \underset{\sim}{y} \in N(A)$, by Definition NSM.

5. Prove that matrix multiplication distributes across matrix addition. That is, if A is an $m \times n$ matrix and B and C are $n \times p$ matrices, then $A(B + C) = AB + BC$. (15 points)

For $1 \le i \le m$, $1 \le j \le p$,

$$[A(B+C)]_{ij} = \sum_{k=1}^{n} [A]_{ik} [B+C]_{kj} \quad \text{Theorem EMP}$$

$$= \sum_{k=1}^{n} [A]_{ik} ([B]_{kj} + [C]_{kj}) \quad \text{Defn MA}$$

$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + [A]_{ik} [C]_{kj} \quad \text{Property DCN}$$

$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + \sum_{k=1}^{n} [A]_{ik} [C]_{kj} \quad \text{Property CACN}$$

$$= [AB]_{ij} + [AC]_{ij} \quad \text{Theorem EMP}$$

$$= [AB + AC]_{ij} \quad \text{Defn MA}$$

So by Defn ME, $A(B+C) = AB + AC$.