Algebras

A General Survey

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What is an Algebra?

Definition: An Algebra, A, is a set, S, under a set of operations.

Definition: An n-ary operation on a set, S, f, takes n elements of S, $(a_1, a_2, ..., a_n)$, to a single element of S, b, denoted,

$$f(a_1, a_2, ..., a_n) = b$$

Familiar Examples

Groups

A group is a set, S, with a single binary operation, $\cdot$, and an inverse (or unary operation), x^{-1}.

The operations on a set are often described by some identities.

The identities describing the binary operation of a group are:

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$$
$$x \cdot 1 \approx 1 \cdot x \approx x$$
$$x \cdot x^{-1} \approx x^{-1} \cdot x \approx 1$$

Familiar Examples

Rings

A ring is a set, S, with:

- two binary operations, addition $(+)$ and multiplication $()$.
- an identity element (or nullary operation), 0, associated with addition
- an additive inverse $(-)$ which can be considered a unary operation

Additional Structure

By including more operations and identities, we can give more structure to our algebra.

In the case of groups, we can include the identity,

$$x \cdot y \approx y \cdot x$$

to form an abelian group.

In the case of rings, we can include

- another nullary operation, 1, which serves the purpose of the identity for multiplication.
- an identity $x \cdot y \approx y \cdot x$

to obtain an Integral Domain (if we assume no zero divisors). We can also include

- another unary operation, x^{-1}, which is our multiplicative inverse

to obtain a field.

Less Structure

In the same way that adding structure can bring us to new algebras, we can also remove operations and identities to obtain new algebras.

Starting with a group, if we remove:

- ▶ the inverse

we obtain an algebra known as a monoid.

- ▶ the identity

we obtain a semigroup.

The set of positive integers with addition is an example of a semigroup.

Boolean Algebra

A Boolean algebra is a set, S, with
- two binary operations, join ($\vee$) and meet ($\wedge$)
- one unary operation, complement ($'$)
- two nullary operations, largest element (I) and smallest element (O)

satisfying the following relations:

$$x \wedge O = O$$
$$x \vee I = I$$
$$x \vee x' = I$$
$$x \wedge x' = O$$

This is a familiar algebra to us however we will introduce an algebra that seems unrelated but reduces to Boolean algebra in special cases.

Ternary Boolean Algebra

We define this ternary Boolean algebra as a set, S, with a ternary operation which we will denote

$$a^b c \text{ for } a, b, c \in S$$

as well as the complement operation from the traditional Boolean algebra.

The ternary operation obeys the following relations:

$$a^b(c^d e) = (a^b c)^d(a^b e)$$
$$a^b b = b^b a = b$$
$$a^b b' = b'^b a = a$$

Ternary Boolean Algebra

Every statement on this page can be proven with little difficulty using the previously stated relations and any statement which sits above it on this page, though the proofs will be omitted.

- ► Every $a \in S$ has a unique complement $a' \in S$.
- ► For all $a \in S$, $(a')' = a$.
- ► The idempotent property holds, $a^b a = a$ for all $a, b \in S$.
- ► Associativity holds, $a^b(c^b d) = (a^b c)^b d$ for all $a, b, c, d \in S$.
- ► For all $a, b \in S$, $a^b a' = b$.
- ► Commutativity holds such that $a^b c = a^c b = b^c a$

Ternary Boolean Algebra

The reason this algebra is called a ternary boolean algebra is that if we allow $p \in S$ to be fixed, we can obtain operations that behave the same way that the binary operations, meet and join, behave in the traditional Boolean algebra. These relationships are

$$a \wedge b \approx a^p b$$
$$a \vee b \approx a^{p'} b$$

This new associated Boolean algebra is denoted $B(p)$ and p serves as the largest element and p' serves as the smallest element.

Polyadic Groups

Whereas a groups is a set with a single binary operation, we can obtain related algebras by allowing the operation to have an arbitrary arity. We call these algebras polyadic groups (or n-ary groups) and we define them as a set, S and an n-ary operation denoted

$$f(a_1, a_2, ..., a_n) = b \text{ where } a_1, ..., a_n, b \in S$$

Polyadic Groups

The *n*-ary operation possesses properties similar to the binary operation of the standard group. For example the operation is associative,

$$f(f(a_1...a_m)a_{m+1}...a_{2m-1}) = f(a_1 f(a_2...a_{m+1})a_{m+2}...a_{2m-1})$$
$$= ...$$
$$= f(a_1...f(a_m...a_{2m-1}))$$

Polyadic Groups

There is identity, however an identity is an ordered sequence of $n-1$ elements of S where n is the arity of the operation, such that if $(a_1, ..., a_{n-1})$ is an identity then

$$f(a_1, ..., a_{n-1}, b) = b.$$

There is also inverse and like identity, it is not generally a single element but rather an ordered sequence of elements. For an ordered sequence of m elements, its inverse must be a sequence of p elements such that $m + p = n - 2$.

Polyadic Groups

The operation of a polyadic groups is defined so that in the case of an 2-adic operation, the polyadic group is simply a traditional group.

Isomorphic Algebras

Definition: Consider two algebras A and B with operations of identical arity. If there exists a function $\alpha : A \to B$ such that for each n-ary operation of A, f^A, α is one-to-one and onto for an operation of B, f^B, satisfying

$$\alpha f^A(a_1, ..., a_n) = f^B(\alpha a_1, ..., \alpha a_n) \text{ for } a_1, ..., a_n \in A,$$

then A and B are isomorphic algebras and α is an isomorphism.

References

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