Quaternion Algebras

Properties and Applications

Rob Eimerl

[1]Department of Mathematics
University of Puget Sound

May 5th, 2015

Quaternion Algebra

- **Definition:** Let F be a field and A be a vector space over a over F with an additional operation $(*)$ from $A \times A$ to A. Then A is an *algebra* over F, if the following expressions hold for any three elements $x, y, z \in F$, and any $a, b \in F$:
  1. Right Distributivity: $(x + y) * z = x * z + y * z$
  2. Left Distributivity: x*(y+z) = (x * y) + (x * z)
  3. Compatability with Scalers: (ax)*(by) = (ab)(x * y)
- **Definition:** A *quaternion algebra* is a 4-dimensional algebra over a field F with a basis $\{1, i, j, k\}$ such that

$$i^2 = a, j^2 = b, ij = -ji = k$$

  for some $a, b \in F^x$. F^x is the set of units in F.
- For $q \in \left(\frac{a,b}{F}\right)$, $q = \alpha + \beta i + \gamma j + \delta k$, where $\alpha, \beta, \gamma, \delta \in F$

Existence of Quaternion Algebras

Theorem 1: Let $a, b \in F^{\times}$, then $\left(\frac{a,b}{F}\right)$ exists.

Proof.

Grab α, β in an algebra E of F such that $\alpha^2 = a$ and $\beta^2 = -b$. Let

$$i = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \; j = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

Then

$$i^2 = a, j^2 = b, ij = \begin{pmatrix} 0 & \alpha\beta \\ \alpha\beta & 0 \end{pmatrix} = -ji.$$

Since $\{I_2, i, j, ij\}$ is linearly independent over E it is also linearly independent over F. Therefore F-span of $\{I_2, i, j, ij\}$ is a 4-dimensional algebra $\mathbb{Q}$ over F, and $\mathbb{Q} = \left(\frac{a,b}{F}\right)$

Associated Quantities of Quaternion Algebras

Pure Quaternions: Let $\{1, i, j, k\}$ be a standard basis for a quaternion algebra $\mathbb{Q}$. The elements in the subspace $\mathbb{Q}_0$ spanned by i, j and k are called the pure quaternions of $\mathbb{Q}$.

Preposition 1: A nonzero element $x \in \mathbb{Q}$ is a pure quaternion if and only if $x \notin F$ and $x^2 \in F$.

Proof.

$(\Rightarrow)$

Let $\{1, i, j, k\}$ be a standard basis for $\mathbb{Q} = \left(\frac{a,b}{F}\right)$. Let x be a nonzero element in $\mathbb{Q}$. We can write $x = a_0 + a_1 i + a_2 j + a_3 k$ with $a_l \in F$ for all l. Then

$$x^2 = (a_0^2 + aa_1^2 + ba_2^2 - (aba_3^2)) + 2a_0(a_1 i + a_2 j + a_3 k)$$

If x is in the F-space spanned by i, j and k, then $a_0 = 0$ and hence $x \notin F$ but $x^2 \in F$.

Quantities Cont.

($\Leftarrow$)
Suppose that $x \notin F$ and $x^2 \in F$. Then one of a_1, a_2, and a_3 is nonzero, and $a_0 = 0$. Thus x is a pure quaternion.
This leads to the idea of a conjugate in $\mathbb{Q}$.
Quaternion Conjugate: $\bar{q} = a - \mathbb{Q}_0 = a - (bi + cj + dk)$

Norm of Quaternion Algebra

Norm: $N : \frac{\alpha,\beta}{F} \to F$, such that

$$N(q) = \bar{q}q = q\bar{q} = a^2 + (-\alpha b^2) + (\beta c^2) + \alpha\beta d^2$$

Norm form: Coefficients of $N(q)$ expressed $< 1, \alpha, \beta, \alpha\beta >$
It should be clear that whenever $N(q) \neq 0$ the element q is invertible, $q^{-1} = \frac{1}{N(q)}\bar{q}$. Indeed q is invertible if and only if $N(q) \neq 0$.
This leads us to our second theorem of Quaternion Algebras.

Quaternion Division Algebras

Theorem 2: The quaternion algebra $\frac{a,b}{F}$ is a division algebra if and only if $N(q) = 0$ implies $q = 0$.

Proof.

$(\Rightarrow)$

Let $\mathbb{Q} = \left(\frac{a,b}{F}\right)$ be a division algebra.

Grab $q \in \mathbb{Q}$ if $N(q) = q\bar{q} = 0$, then $q = 0$, or $\bar{q} = 0$, either of which means $q = 0$.

$(\Leftarrow)$

If $N(q) \neq 0$, $q^{-1} = \frac{\bar{q}}{N(q)}$. Then $N(q) = 0 \rightarrow q = 0$, and any non-zero element in $\left(\frac{a,b}{F}\right)$ is invertible. Thus $\left(\frac{a,b}{F}\right)$ is a division algebra.

Isomorphisms of Quaternion Algebras

The Norm Form of a quaternion algebra also provides a way to test whether two quaternion algebras are isomorphic.
Two forms, $Q_1 : V_1 \to F$ and $Q_2 : V_2 \to F$ are isometric if their exists a vector space isomorphism $\phi : V_1 \to V_2$, such that $Q_2(\phi(x)) = Q_1(x)$ for all $x \in V_1$. [4]
Theorem 3: Given 2 quaternion algebras $\mathbb{Q} = \left(\frac{a,b}{F}\right)$, and $\mathbb{Q}' = \left(\frac{a',b'}{F}\right)$; the following are equivalent:

1. $\mathbb{Q}$ and $\mathbb{Q}'$ are isomorphic.
2. The norm forms of $\mathbb{Q}$ and $\mathbb{Q}'$ are isometric.
3. The norm forms of $\mathbb{Q}_0$ and $\mathbb{Q}'_0$ are isometric.

Isomorphisms Cont.

Proof.

(1 $\Rightarrow$ 2) Since ϕ is an F-algebra isomorphism, by Proposition 1 we have

$$\begin{aligned} v \in \mathbb{Q}_{\nvdash} &\Leftrightarrow v \notin F, v^2 \in F \\ &\Leftrightarrow \phi(v) \notin F, \phi(v)^2 \in F \\ &\Leftrightarrow \phi(v) \in \mathbb{Q}'_0 \end{aligned} \tag{1}$$

If $x = \alpha + x_0$ where $\alpha \in F$ and $x_0 \in \mathbb{Q}_0$, then $\bar{x} = \alpha x_0$, and hence $\phi(x) = \alpha + \phi(x_0)$ and $\phi(x) = \alpha\phi(x_0)$.
Since $\phi(x_0) \in \mathbb{Q}'_0$, we have $\overline{\phi(x)} = \phi(\bar{x})$. Thus

$$N(\phi(x)) = \phi(x)\overline{\phi(x)} = \phi(x)\phi(\bar{x}) = \phi(N(x)) = N(x)$$

so ϕ is an isometry from $\mathbb{Q}$ to $\mathbb{Q}'_0$.
The proof of the remaining 2 equivalences can be found in [10].

Characteristics of Quaternion Algebras

Theorem 4: A quaternion algebra over F is central simple, that is, its center is F and it does not have any nonzero proper two-sided ideal.

Proof.

Let $\mathbb{Q}$ be a quaternion algebra over F, and $\{1, i, j, k\}$ be a standard basis of $\mathbb{Q}$ over F. Consider an element $x = \alpha + \beta i + \gamma j + \delta k$ in the center of $\mathbb{Q}$, where $\alpha, \beta, \gamma, \delta \in F$. Then

$$\begin{aligned} x\mathbb{Q} &= x\mathbb{Q}, \text{ for all } x \in \mathbb{Q} \\ \Rightarrow 0 &= jx - xj \\ &= j(\alpha + \beta i + \gamma j + \delta k) - (\alpha + \beta i + \gamma j + \delta k)j \\ &= 2k(\beta + \delta j). \end{aligned} \tag{2}$$

Since k is invertible in $\mathbb{Q}$, it must be that $\beta = \delta = 0$. Similarly, it can be shown $\gamma = 0$. Hence $x \in F$.

Characteristics Cont.

Proof cont.

We need to show that any nonzero two-sided ideal I is $\mathbb{Q}$ itself. It is sufficient to show that I contains a nonzero element of F. Take a nonzero element $y = a + bi + cj + dk \in I$, where $a, b, c, d \in F$. We may assume that one of b, c and d is nonzero.

By replacing y by one of iy, jy and ky, we may further assume that $I \neq 0$. Since $yj - jy \in I$ and $2k$ is invertible in $\mathbb{Q}$, we see that $b + dj$, and hence $bi + dk$, are in I. This shows that $a + cj$ is in a. Similarly, $a + bi$ and $a + dk$ are also in I. Therefore,

$$-2a = y(a + bi)(a + cj)(a + dk) \neq 0 \text{ is from } F \text{ and resides in } I$$

Thus $I = \mathbb{Q}$

Polynomials in $\mathbb{Q}$

The Fundamental Theorem of Algebra states: Any polynomial of degree n with coefficients in any field F can have at most n roots in F. For polynomials with coefficients from $\mathbb{Q}$ the situation is somewhat different.
Due to the lack of commutativity in $\mathbb{Q}$ polynomials become commensurately more complicated, in just degree two we may have terms like $ax^2, xax, x^2a, axbx$ all of which are distinct in $\mathbb{Q}$.
However, there is a Fundamental Theorem of Algebra for $\mathbb{Q}$, which says that if the polynomial has only one term of highest degree then there exists a root in $\mathbb{Q}$.
This can be pushed further for division algebras using the Wedderburn Factorization Theorem for polynomials over division algebras.

Wedderburn Factorization Theorem

Theorem 5: Let D be a division ring with center F and let p(x) be an irreducible monic polynomial of degree n with coefficients from the field F. If there exists $d \in D$ such that $p(d) = 0$ then we can write

$$p(x) = (t - d_1)(t - d_2)(t - d_3)(t - d_n)$$

and each d_i is conjugate to d_1; there exist nonzero elements $s_i \in D$, such that $d_i = s_i d s_i^{-1}$ for $1 \leq i \leq n$.
This theorem says that if the polynomial has one root in D then it factorizes completely as a product of linear factors over D!

Extensions of Quaternion Algebras

It is possible to describe an algebra as an extension of a smaller algebra.
The process used for building quaternion algebras is known as Cayley-Dickson Doubling. It is a way of extending an algebra A to a new algebra, $KD(A)$, and preserving all operations (addition, scalar multiplication, element multiplication and the norm), such that A is a subalgebra of $KD(A)$. If A has a unity element Θ then so does $KD(A)$ and the extension can be expressed;

$$KD(A, \Theta) = A \oplus A\mu$$

where μ is a root of unity.
Any extension is a 2 degree extension over the preceding algebra.

Frobenius

The mathematician Frobenius took this idea of subalgebras and found an incredible result about Real Division Algebras.

Theorem 6: Suppose A is an algebra with unit over the field R of reals. Assume that the algebra A is without divisors of zero. If each element $x \in A$ is algebraic with respect to the field $\mathbb{R}$ then the algebra is isomorphic with one of the classical division algebras $\mathbb{R}, \mathbb{C},$ or $\mathbb{Q}$.

We have just classified all real division algebras!

Applications of Quaternion Algebras

The are a myriad of uses for quaternion algebras including:

- Group Theory: The quaternions form an order 8 subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$.
- Number Theory: The mathematician Hurwitz introduced the ring of integral quaternions. This construction was used to prove Lagrange's theorem, that every positive integer is a sum of at most four squares.
- Rotations: Quaternions can describe rotations in 3-dimensional space. Traditionally rotations are considered compositions of rotations around the Cartesian coordinate axes by angles ψ, ϕ and θ. However, Euler proved that a general rotation of a rigid object can be described as a single rotation about some fixed vector. Given $v = [l, m, n]$ over $\mathbb{R}^3$ then a rotation by an angle θ about v is given by

$$L_q(v) = qvq^* \text{ where } q = [\cos\frac{\theta}{2}, l\sin\frac{\theta}{2}, m\sin\frac{\theta}{2}, n\sin\frac{\theta}{2}]$$

Applications of Quaternion Algebras Cont.

- Computer Graphics: The quaternions on the other hand generate a more realistic animation. A technique which is currently gaining favor is called spherical linear interpolation (SLERP) and uses the fact that the set of all unit quaternions form a unit sphere. By representing the quaternions of key frames as points on the unit sphere, a SLERP defines the intermediate sequence of rotations as a path along the great circle between the two points on the sphere.
- Physics: The quaternions have found use in a wide variety of research.
  - They can be used to express the Lorentz Transform making them useful for work on Special and General Relativity.
  - Their properties as generators of rotation make them incredibly useful for Newtonian Mechanics, scattering experiments such as crystallography, and quantum mechanics (particle spin is an emergent property of the mathematics).

References I

- CWoo,Cayley-Dickson Construction. March 21, 2013 http://planetmath.org/sites/default/files/texpdf/36586.pdf
- Baez, John C. The Octonions. Department of Mathematucs, UC Riverside. May 16, 2001
- Boggdan, Victor M. On Frobenius, Mazur, and Gelfand-Mazur Theorems on Division Algebras. Department of Mathematics, CUA, Washington DC. 2006
- Lewis, David W. Quaternion Algebras and the Algebraic Legacy of Hamiltons Quaternions. Irish Math. Soc. Bulletin 57 (2006), pp 41-64.
- Mukundan, R. Quaternions: From Classical Mechanics to Computer Graphics, and Beyond. Department of Computer Science, University of Canterbury, Christchurch, New Zealand.

References II

- Sudbery, A. Quaternionic Analysis Math. Proc. Camb. Phil. Soc. (1979), **85**, 199-225
- Eilenberg, Samuel Niven, Ivan. The Fundamental Theorem of Algebra For Quaternions. University of Michigan and Purdue University. 1939.
- Hazewinkel, Micheal, Nadiya Gubareni, V.V. Kirichenko. Algebra, Rings and Modules part 1. Springer Science Business Media, Oct 1, 2004
- Girard, P R. The Quaternion Group and Modern Physics Eur. J. Phys. **5** 25, 1984
- Arithmetic of Quaternion Algebra 2012. http://wkchan.web.wesleyan.edu/quaternion-2012.pdf

References III

Zi Yang Sham. Quaternion Algebras and Quadratic Forms
University of Waterloo. Waterloo, Ontario, Canada, 2008