The Quaternions and their Applications

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Abstract

This paper is an attempt to summarize Quaternion Algebras. The first part looks at their origins and certain properties of these algebras are examined, from the roots of Polynomials over the Quaternions to how to construct a quaternion algebra and Frobenius' theorem.
The second part of this paper looks at applications of quaternion algebras. For the sake of brevity proofs are omitted from this paper but are cited

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1 Introduction

In late 1843 the great Irish mathematician William Hamilton was struggling over a problem that had confounded him for nearly a decade, how to multiply triplets. In 1835 he had developed techniques that allowed complex numbers to be treated as pairs of real numbers ($a + bia, b \in$). Intrigued by the relation between complex numbers (C) and 2-dimensional geometry, Hamilton tried for years to develop an analogous 3-dimensional algebra.
To appreciate the problem consider a vector space (V), with the basis a, b, c, and add the additional operation * (vector multiplication that distributes over addition). If we take $v = a + b + c$ (the simplest linear combination of every basis vector) by closure v^2 should also be an element of V. However, when we compute v^2 we find

$$\begin{aligned} v^2 &= v(v) = (a + b + c) * (a + b + c) \\ &= a(a + b + c) + b(a + b + c) + c(a + b + c) \\ &= a^2 + ab + ac + ba + b^2 + bc + ca + cb + c^2 \\ &= (a^2 + b^2 + c^2) + ab + ac + ba + bc + ca + cb \end{aligned} \tag{1}$$

This is a hopelessly confusing result, that doesn't make sense until multiplication of the basis vectors is defined. But how should such an operation be defined? We need six products that preserve closure and the other properties of V.
On the 16th of October, Hamilton had a brain-wave. While walking to a meeting of the Royal Irish Academy, he felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, k exactly as I have used them ever since. He famously carved these equations into the stone of Brougham Bridge in Dublin

$$i^2 = j^2 = k^2 = ikj = -1,$$

these first relations quickly led to secondary ones:

$$ij = -ji = k, ik = -ki = j, jk = -kj = i.$$

Hamilton called $1, i, j, k$ the quaternions and with them built the worlds first non-commutative algebra1, opening the door to a wide range of new algebraic structures and ideas, and in a very real way to development of Abstract Algebra as field of study[4].

2 Quaternion Algebra

2.1 Preliminaries

Algebra: Let F be a field, and let A be a vector space over F with an additional binary operation ($*$) from $A \times A$ to A. Then A is an algebra over F if the following expressions hold for any three elements $x, y, z \in A$, and for a, b ∈F: [8]

1. Right distributivity: $(x + y) * z = x * z + y * z$

2. Left distributivity: $x * (y + z) = x * y + x * z$

3. Compatibility with scalars: $(ax) * (by) = (ab)(x * y)$.

2.2 The Real Quaternions (H)

The first algebraic structure Hamilton built was the Real Quaternions, H:

$$H = \{q = a + bi + cj + dk : a, b, c, d \in R; \{1, i, j, k\} \in Q\}$$

H is a 4-dimensional vector space over R with the standard forms of addition and scalar multiplication, that is then boosted into a ring by defining quaternion multiplication (indicated by juxtaposition) via $i^2 = j^2 = 1, ij = ji = k$[4], such that

$$\begin{aligned} q_1 q_2 = &(a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) \\ &+ (a_1 b_2 - a_2 b_1 - c_1 d_2 - d_1 c_2)i \\ &+ (a_1 c_2 - b_1 d_2 + c_1 a_2 - d_1 b_2)j \\ &+ (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)k \end{aligned} \tag{2}$$

2.3 Generalized Quaternion Algebra (Q)

This leads to a generalized Quaternion Algebra (Q) by replacing with any field F, and redefining basis multiplication. A quaternion algebra over a field F is written as $Q = \left(\frac{\alpha,\beta}{F}\right)$ and elements of Q have the form

$$q = a + bi + cj + dk$$

Quaternion multiplication is defined by α and β (nonzero elements of F, $a = b$ not excluded) such that the new multiplication rules are

$$i^2 = a, j^2 = bij = -ji = k$$

At this point it is worth defining certain quantities associated with quaternion algebras [4] Let $q \in \frac{a,b}{F}, q = a + bi + cj + dk$

Quaternion Conjugate: $\bar{q} = a + (-b)i + (-c)j + (-d)k$

Complex Conjugate: $q^* = a^* + (-b^*)i + (-c^*)j + (-d^*)k$

Norm: $N : \frac{a,b}{F} \to F$ such that

$$N(q) = \bar{q}q = q\bar{q} = a^2 + (-\alpha b^2) + (\beta c^2) + \alpha\beta d^2$$

Norm form: Coefficients of $N(q)$ expressed $< 1, \alpha, \beta, \alpha\beta >$

It should be clear that whenever $N(q) \neq 0$ the element q is invertible, $q^{-1} = \frac{1}{N(q)}\bar{q}$. Indeed q is invertible if and only if $N(q) \neq 0$.
This leads us to our first theorem of Quaternion Algebras.
Theorem 1: The quaternion algebra $\frac{a,b}{F}$ is a division algebra if and only if N(q) = 0 implies q = 0 [4].

The Norm form also gives a way to test whether two quaternion algebras are isomorphic.
Theorem 2: Quaternion Algebras are isomorphic as algebras if and only if their norm forms are isometric.

Two forms, $Q_1 : V_1 \to F$ and $Q_2 : V_2 \to F$ are isometric if their exists a vector space isomorphism $\phi : V_1 \to V_2$, such that $Q_2(\phi(x)) = Q_1(x)$ for all $x \in V_1$. [4]
Example:

$$\frac{a,b}{F} \cong \frac{b,a}{F},$$

their norm forms $< 1, a, b, ab >$ and $< 1, b, a, ab >$ are isometric.

2.4 Polynomials in H

The subject of algebra began as attempts to solve various equations,particularly polynomial equations. The Fundamental Theorem of Algebra states: Any polynomial of degree n with coefficients in any field F can have at most n roots in F. For polynomials with coefficients from H the situation is somewhat different. Due to the lack of commutativity polynomials become commensurately more complicated, in just degree two we may have terms like $ax^2, xax, x^2a, axbx$. However, there is a Fundamental Theorem of Algebra for H which says that if the polynomial has only one term of highest degree then there exists a root in H[7]. In the context of solving polynomial equations we should mention the Wedderburn Factorization Theorem for polynomials over division algebras.

Wedderburn Factorization Theorem[4]
Let D be a division ring with center F and let p(x) be an irreducible monic polynomial of degree n with coefficients from the field F. If there exists $d \in D$ such that $p(d) = 0$ then we can write $p(x) = (td_1)(td_2)(td_3)(td_n)$ and each d_i is conjugate to d_1; there exist

nonzero elements $s_i \in D$, such that $d_i = s_i d s_i^1$ for each i.
This theorem says that if the polynomial has one root in D then it factorizes completely as a product of linear factors over D. This is a major improvement compared to the situation in field theory where only one root of a polynomial is guaranteed in a given field.

2.5 Extensions of Quaternion Algebras

In December 1843 John Graves extended the quaternions and discovered the octonions O, an 8-dimensional algebra over R which is both noncommutative and nonassociative[2]. The process of going from R to C, from C to H, and from H to O, known as Cayley-Dickson Doubling, is a way of extending an algebra A to a new algebra, KD(A), and preserving all operations (addition, scalar multiplication, element multiplication and the norm), such that A is a subalgebra of KD(A). If A has a unity element Θ then so does $KD(A)$ and the extension can be expressed;

$$KD(A, \Theta) = A \oplus A\mu$$

where μ is a root of unity. Each extension is a 2 degree extension of the preceding algebra.[1]

Example: Extensions of R

$$\begin{aligned} KD(R) &= R + Ri = a + bi = C \\ KD(C) &= C + Cj = (a + bi) + (c + di)j = Q \end{aligned} \tag{3}$$

Generalizing this example to any field F we can define the Quaternion algebra over F as $KD(KD(F, \Theta_1), \Theta_2)$, where $\Theta_i \in F \setminus \{0\}$

Note that at each stage of the doubling process one of the intrinsic properties of the base algebra is lost. From R to C we lose ordering, from C to H commutativity and from H to O associativity. If we apply the CayleyDickson doubling process to the octonions we obtain a structure called the sedenions, which is a 16-dimensional nonassociative algebra, However the sedenions are not a division ring and not very conducive for working with.

The mathematician Frobenius took this idea of subalgebras and found an interesting resul about Real Division Algebras.[3]

Theorem (Frobenius): Suppose A is an algebra with unit over the field R of reals. Assume that the algebra A is without divisors of zero. If each element $x \in A$ is algebraic with respect to the field R then the algebra is isomorphic with one of the classical division algebras $R, C,$ or Q.

We have just classified all real division algebras, there are additional theorems about Quaternion Algebra that have not been discussed, but at this point I would like to examine some applications.

3 Applications of Quaternion Algebra

There are a myriad of different applications across a wide variety of subjects that use quaternion algebra some of these include:

- Group Theory: The quaternions form an order 8 subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$. We discussed properties of this group in class.
- Number Theory: The mathematician Hurwitz introduced the ring of integral quaternions, a subring of H consisting of all quaternions of the form $a + bi + cj + dk$ where either each of $a, b, c, d \in Z$, or else each of a, b, c, d is congruent to $\frac{1}{2}$ modulo Z. This construction was used to prove Lagrange's theorem, that every positive integer is a sum of at most four squares [4].
- Rotations: Easily the most applicable property of the quaternions is that can describe rotations in 3-dimensional space. Traditionally rotations are considered compositions of rotations around the Cartesian coordinate axes by angles ψ, ϕ and θ. Euler proved that a general rotation of a rigid object can be described as a single rotation about some fixed vector[5]. Given $v = [l, m, n]$ over R^3 then a rotation by an angle θ about v is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} l^2(1-\cos\theta)+\cos\theta & lm(1-\cos\theta)-n\sin\theta & nl(1-\cos\theta)+m\sin\theta \\ lm(1-\cos\theta)+n\sin\theta & m^2(1-\cos\theta)+\cos\theta & mn(1-\cos\theta)-l\sin\theta \\ nl(1-\cos\theta)-m\sin\theta & mn(1-\cos\theta)+l\sin\theta & n^2(1-\cos\theta+\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

  The above transformation can much more compactly expressed as the quaternion product:

$$w = qvq^*$$

  where $w = [0, x', y', z'], v = [0, x, y, z]$, and $q = [\cos\frac{\theta}{2}, l\sin\frac{\theta}{2}, m\sin\frac{\theta}{2}, n\sin\frac{\theta}{2}]$. As a rotation does not affect the magnitude of a vector, q must be a unit quaternion ($N(q) = 1$). Thus we can define the quaternion rotation operator

$$L_q(v) = qvq^*$$

  This ability to completely describe any 3-dimensional rotation has far-reaching consequences when applied to computer graphics, and to physics.

- Computer Graphics: Given two orientations in the 3-dimensions, programmers traditionally used linear interpolation between the corresponding Euler angles to model a rotation. Such an algorithm can cause singularities, and problems such as gimbal

lock (rotation in one axis is momentarily forbidden) may be encountered, which would severely affect the smoothness of the animation.
The quaternions on the other hand generate a more realistic animation. A technique which is currently gaining favor is called spherical linear interpolation (SLERP) and uses the fact that the set of all unit quaternions form a unit sphere. By representing the quaternions of key frames as points on the unit sphere, a SLERP defines the intermediate sequence of rotations as a path along the great circle between the two points on the sphere [5].

- Physics: The quaternions have found use in a wide variety of research.
  - They can be used to express the Lorentz Transform making them useful for work on Special and General Relativity[9].
  - Their properties as generators of rotation make them incredibly useful for Newtonian Mechanics, scattering experiments such as crystallography, and quantum mechanics (Particle spin is an emerge property of the math) [9].

4 Conclusion

Quaternion Algebras are bit of a mixed bag, clearly they are a very interesting and powerful tool for both modeling certain phenomena, and for algebraic study. However, lacking commutativity makes working with quaternion algebras cumbersome. In some cases though they do offer a clear advantage as is the case for 3-dimensional rotations; as they are significantly easier to compute both for humans and machines expressed as quaternions.

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