Math 290 Name: Key Dr. Beezer
Exam 4 Spring 2016
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. You may use Sage to form, manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features of something I can't see).

1. For the matrix A below, compute the dimensions of the null space, the column space, the row space, and the left null space. (15 points)

$$A = \begin{bmatrix} -1 & 4 & 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -2 & 5 & 4 & -3 \\ -2 & 8 & 1 & -3 & 3 & 7 & -8 \\ 2 & -8 & 0 & 3 & -2 & -5 & 7 \end{bmatrix}$$

$n = \#\text{ cols} = 7$
$m = \#\text{ rows} = 4$

$$\xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & -4 & 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r = 3$

Answer: $n(A) = \dim(\mathcal{N}(A)) = n - r = 4$; $r(A) = \dim(\mathcal{C}(A)) = r = 3$; $\dim(\mathcal{R}(A)) = r = 3$; $\dim(\mathcal{L}(A)) = m - r = 1$

2. Compute a basis for the subspace W of the vector space of 2×2 matrices, M_{22}. (15 points)

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, 2a + 5b + 13c - d = 0, a + 3b + 8c - d = 0 \right\}$$

The membership criteria has a homogenous system w/ coeff. matrix

$$\begin{bmatrix} 2 & 5 & 13 & -1 \\ 1 & 3 & 8 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \end{bmatrix}, \text{ so}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c - 2d & -3c + d \\ c & d \end{bmatrix}$$

$$= c \begin{bmatrix} 1 & -3 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$$

which provides a spanning set for W.

Showing that B is linearly independent is straight forward (but necessary).

Answer:
$$B = \left\{ \begin{bmatrix} 1 & -3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

3. The questions on this page ask about properties of subsets of the vector space of polynomials with degree at most 2, P_2. Give complete and detailed explanations of your answers. (40 points)

Note: $\dim P_2 = 2+1 = 3$.

(a) $Q = \{3x^2 - 6x + 2,\ -4x^2 + x + 9\}$. Does Q span P_2?

size $Q = 2 < 3 = \dim P_2$,

so by Theorem G, Q does not span P_2.

(b) $R = \{-x^2 + 2x + 3,\ 4x^2 + 2x - 4,\ x^2 + 5x - 1,\ 3x^2 + 2x - 5\}$. Is R linearly independent in P_2?

size $R = 4 > 3 = \dim P_2$

so by Theorem G, R does not span P_2.

(c) $S = \{x^2 - x + 2,\ -3x^2 + 6x - 8,\ 3x^2 - 5x + 7,\ -4x^2 + 4x - 7\}$. Does S span P_2?

(Theorem G does not apply.)

$\alpha_1(x^2-x+2) + \alpha_2(-3x^2+6x-8) + \alpha_3(3x^2-5x+7) + \alpha_4(-4x^2+4x-7) = ax^2+bx+c$

$\Rightarrow$ system $\left[\begin{array}{cccc|c} 1 & -3 & 3 & -4 & a \\ -1 & 6 & -5 & 4 & b \\ 2 & -8 & 7 & -7 & c \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & \sim \\ 0 & 1 & 0 & -2 & \sim \\ 0 & 0 & 1 & -3 & \sim \end{array}\right]$

always a consistent system so, yes, S does span P_2.

(d) $T = \{2x^2 - x + 1,\ -3x^2 + 1,\ x^2 - 2x + 3,\ -5x^2 + x\}$. Does T span P_2?

We can mimic part (c), or even pick a "random" polynomial $r(x) = x^2+x \in P_2$ with $r(x) \notin \langle T\rangle$:

$\alpha_1(2x^2-x+1) + \alpha_2(-3x^2+1) + \alpha_3(x^2-2x+3) + \alpha_4(-5x^2+x) = x^2+x$

$\Rightarrow$ system $\left[\begin{array}{cccc|c} 2 & -3 & 1 & -5 & 1 \\ -1 & 0 & -2 & 1 & 1 \\ 1 & 1 & 3 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

An inconsistent system, so $r(x) \notin \langle T\rangle$.

4. Suppose that V is a vector space. Prove that $-\mathbf{v} = (-1)\mathbf{v}$ for each $\mathbf{v} \in V$. (15 points)

$$\begin{aligned} \underset{\sim}{0} &= 0\underset{\sim}{v} \\ &= (-1+1)\underset{\sim}{v} \\ &= (-1)\underset{\sim}{v} + 1\underset{\sim}{v} \\ &= (-1)\underset{\sim}{v} + \underset{\sim}{v} \end{aligned}$$

This shows that $(-1)\underset{\sim}{v}$ is the additive inverse of $\underset{\sim}{v}$, that is,

$$-\underset{\sim}{v} = (-1)\underset{\sim}{v}.$$

This is Theorem AISM, see text for a different proof & some relevant reasons.

5. Consider the subsets S and T from P_2, the vector space of polynomials with degree at most 2. Prove that their spans are equal, that is $\langle S \rangle = \langle T \rangle$. (15 points)

$$S = \{1 + x + x^2,\ -2 + x - x^2\} \qquad T = \{-5 + 7x - x^2,\ -4 + 5x - x^2\}$$

We want to prove a set equality. If each vector in T is also in $\langle S \rangle$, then any linear combination of the vectors from T will be in $\langle S \rangle$ (since $\langle S \rangle$ is closed) & then $\langle T \rangle \subseteq \langle S \rangle$.

Use two systems of equations to see that

$$-5 + 7x - x^2 = 3(1 + x + x^2) + 4(-2 + x - x^2)$$

$$-4 + 5x - x^2 = 2(1 + x + x^2) + 3(-2 + x - x^2)$$

(A) Now show $\langle S \rangle \subseteq \langle T \rangle$ by a similar procedure.

OR

(B) Show S & T are both linearly independent, hence bases of $\langle S \rangle$ & $\langle T \rangle$, respectively. Then $\dim \langle T \rangle = 2 = \dim \langle S \rangle$ & Theorem EDYES $\Rightarrow \langle T \rangle = \langle S \rangle$