Quaternion Algebras

Edgar Elliott

May 1, 2016

The Hamiltonian Quaternions

The Hamiltonion quaternions $\mathbb{H}$ are a system of numbers devised by William Hamilton in 1843 to describe three dimensional rotations.

- $q = a + bi + cj + dk$ where $i^2 = j^2 = k^2 = ijk = -1$
- non-abelian multiplication

Conjugation and Norms

- Conjugation in the Hamiltonian quaternions is defined as follows: if $q = a + bi + cj + dk$ then $\overline{q} = a - bi - cj - dk$.
- The norm is defined by $N(q) = q\overline{q} = \overline{q}q = a^2 + b^2 + c^2 + d^2$.

Properties

Some important properties of the conjugate and norm.

- $\overline{\overline{q}} = q$
- $\overline{q_1 + q_2} = \overline{q_1} + \overline{q_2}$
- $\overline{q_1 q_2} = \overline{q_2}\ \overline{q_1}$
- Elements with nonzero norms have multiplicative inverses of the form $\frac{\overline{q}}{N(q)}$.
- The norm preserves multiplication

$$N(q_1 q_2) = q_1 q_2 \overline{q_1 q_2} = q_1 q_2 \overline{q_2}\ \overline{q_1} = q_1 N(q_2) \overline{q_1}$$

$$= N(q_2) q_1 \overline{q_1} = N(q_2) N(q_1)$$

Definition of an Algebra

An algebra over a field is a vector space over that field together with a notion of vector multiplication.

Generalizing the Quaternions

The Hamiltonian quaternions become a prototype for the more general class of quaternion algebras over fields. Defined as follows:

- A quaternion algebra $(a, b)_F$ with $a, b \in F$ is defined by $\{x_0 + x_1 i + x_2 j + x_3 k | i^2 = a, j^2 = b, ij = k = -ji, x_i \in F\}$.
- Under this definition we can see that $\mathbb{H} = (-1, -1)_\mathbb{R}$ since

$$k^2 = (ij)^2 = ijij = -iijj = -(-1)(-1) = -1$$

- Note: We will always assume that $char(F) \neq 2$.

Generalizing Conjugates and Norms

- Conjugation works the same $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$
- The Norm is defined as $N(q) = \overline{q}q = q\overline{q} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$, it still preserves multiplication.
- Inverse elements are still defined as $\frac{\overline{q}}{N(q)}$ for elements with nonzero norms.

The Split Quaternions

The split-quaternions are the quaternion algebra $(1, -1)_{\mathbb{R}}$.

- Allows for zero divisors and nonzero elements with zero norms

$$(1 + i)(1 - i) = 1 + i - i - 1 = 0$$

Isomorphisms of quaternion Algebras

An isomorphism between quaternion algebras is a ring isomorphism that fixes the "scalar term".

- For example:

$$1 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i \to \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, j \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, k \to \begin{bmatrix} 0 & -1 \\ a & 0 \end{bmatrix}$$

  is an isomorphism from any quaternion algebra $(a, 1)_F$ to $M_2(F)$ the algebra of 2×2 matrices over F.

Quaternionic Bases

A quaternionic basis is a set $\{1, e_1, e_2, e_1e_2\}$ where $e_1^2 \in F$, $e_2^2 \in F$, $e_1^2, e_2^2 \neq 0$, and $e_1e_2 = -e_2e_1$.
Isomorphisms between quaternion algebras can be determined through the construction of quaternionic bases. If you can construct bases in two algebras such that the values of e_1^2 and e_2^2 are equal, then those algebras are isomorphic to one another.

- This shows tha $(a, b)_F$, $(b, a)_F$, $(a, -ab)_F$ and all similar permutations of a, b, and$-ab$ produce isomorphic algebras.

Important Categories of Isomorphism

- $(a, b^2)_F \cong M_2(F)$
- Since an isomorphism exists:

$$1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i \rightarrow \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, j \rightarrow \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}, k \rightarrow \begin{bmatrix} 0 & -b \\ ab & 0 \end{bmatrix}$$

Important Categories of Isomorphism Cont.

$(a,b)_F \cong M_2(F)$ if $b = x^2 - ay^2$ for $x, y \in F$
To show this we construct a basis $\{1, i, jx + ky, (i)(jx + ky)\}$, this is clearly a basis of $(a,b)_F$ and since

$$(jx + ky)^2 = j^2x^2 + jkxy + kjxy + k^2y^2$$

$$= bx^2 - aby^2 = b(x^2 - ay^2) = b^2$$

It is also a basis of $(a, b^2)_F$ so
$(a, x^2 - ay^2)_F \cong (a, b^2)_F \cong M_2(F)$.

The Norm Subgroup

Elements of a field of the form $x^2 - ay^2$ for a given a form a group under multiplication known as the norm subgroup associated to a or N_a.

- $1 = 1^2 - a0^2$
- $(x^2 - ay^2)(w^2 - az^2) = (xw + ayz)^2 - a(xz + wy)^2$
- $$\frac{1}{x^2 + ay^2} = \frac{x^2 + ay^2}{(x^2 + ay^2)^2} = {\frac{x}{x^2 + ay^2}}^2 - a{\frac{y}{x^2 + ay^2}}^2$$

Real Quaternion Algebras

Theorem: There are only two distinct quaternion algebras over $\mathbb{R}$ which are $\mathbb{H}$ and $M_2(\mathbb{R})$.
Proof:

- Given $(a,b)_{\mathbb{R}}$ if $a,b<0$ then we can construct a basis $\{1,\sqrt{-a}i,\sqrt{-b}j,\sqrt{ab}ij\}$ in $\mathbb{H}$ which forms a basis of $(a,b)_{\mathbb{R}}$ indicating the existence of an isomorphism.
- If $a>0, b<0$ WLOG, we can construct a basis $\{1,\sqrt{a}i,\sqrt{-b}j,\sqrt{-ab}ij\}$ in the $(1,-1)_{\mathbb{R}}$ which forms a basis of $(a,b)_F$ indicating the existence of an isomorphism with the split-quaternions and therefore $M_2(F)$.

Complex Quaternion Algebras

Theorem: There is only one quaternion algebra over $\mathbb{C}$, which is $M_2(\mathbb{C})$.
Proof:

- We've shown that $(a, b^2)_F \cong M_2(F)$. We can find always find a $c \in \mathbb{C}$ such that $c^2 = b$, therefore $(a, b)_{\mathbb{C}} \cong (a, c^2)_{\mathbb{C}} \cong M_2(\mathbb{C})$.

Categorizing Quaternion Algebras

Theorem: All quaternion algebras that are not division rings are isomorphic to $M_2(F)$

Proof: Take a quaternion algebra $A = (a, b)_F$

- If $a = c^2$ or $b = c^2$ for some $c \in F$ then $A \cong M_2(F)$, now assume neither a nor b are squares.
- If A isn't a division ring then there must be some nonzero element without a multiplicative inverse. We will show that $b = x^2 - ay^2$ and therefore $(a, b)_F \cong M_2(F)$.

Categorizing Quaternion Algebras Cont.

- The only elements without inverses are those with $N(q) = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2 = 0$
- $x_1^2 - ax_2^2 = b(x_3^2 - ax_4^2)$
- $x_3^2 - ax_4^2 \neq 0$ since either $x_3 = x_4 = 0$ or $a = \frac{x_3^2}{x_4^2}$. If $x_3 = x_4 = 0$ then either $x_1 = x_2 = 0$ or $a = \frac{x_1^2}{x_2^2}$. All of which are contradictions.
- So $b = \frac{x_1^2 - ax_2^2}{x_3^2 - ax_4^2}$, therefore $b = x^2 - ay^2$ by closure of N_a so $A \cong M_2(F)$.

Rational Quaternion Algebras

It can be shown that there are infinite distinct quaternion algebras over $\mathbb{Q}$. By the previous theorem all but $M_2(\mathbb{Q})$ must be division rings.

The Octonions

The octonions are another set of numbers, discovered independantly by John T. Graves and Arthur Cayley in 1843, which are of the form:

$$o = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7$$

- Multiplication neither commutative nor associative
- Obeys the Moufang Identity $(z(x(zy))) = (((zx)z)y)$, weaker than associativity but behaves similarly.
- Conjugation behaves the same.
- Norm still preserves multiplication.

The Fano Plane

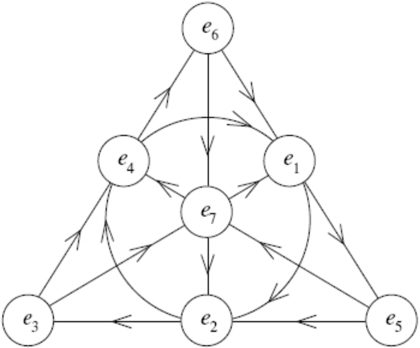

Figure: The Fano plane

Generalizing Octonion Algebras

Much as quaternion algebras can be described by $(a, b)_F$ octonion algebras can be described by three of their seven in the form $(a, b, c)_F$.

- $(-1, -1, -1)_{\mathbb{R}}$ are Graves' octonions
- $(1, 1, 1)_{\mathbb{R}}$ are the split-octonions
- these are the only two octonion algebras over $\mathbb{R}$

Zorn Vector-Matrices

Unlike the quaternions, octonions and by extension octonion algebras cannot be expressed as matrices since matrix multiplication is associative. German mathematician Max August Zorn created a system called a vector-matrix algebra which could be used to describe them.

$$\begin{bmatrix} a & \mathbf{u} \\ \mathbf{v} & b \end{bmatrix} \begin{bmatrix} c & \mathbf{w} \\ \mathbf{x} & d \end{bmatrix} = \begin{bmatrix} ac + \mathbf{u} \cdot \mathbf{x} & a\mathbf{w} + d\mathbf{u} - \mathbf{v} \times \mathbf{x} \\ c\mathbf{v} + b\mathbf{x} + \mathbf{u} \times \mathbf{w} & bd + \mathbf{v} \cdot \mathbf{w} \end{bmatrix}$$

Other Notes on Octonion Algebras

- Two complex elements that are not scalar multiples of one-another generate a quaternion subalgebra.
- Information about isomorphisms is less readily available, it's clear that some of the same principles apply but with added difficulty.
- Sedenion algebras (16-dimensional) and above cease being composition algebras.

Questions?