Partial, Total, and Lattice Orders in Group Theory

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Orders

- A relation on a set X is a subset of $X \times X$
- A *partial order* is reflexive, transitive, and antisymmetric
- A *total order* is dichotomous (either $x \preceq y$ or $y \preceq$ for all $x, y \in X$)
- In a *lattice-order*, every pair or elements has a least upper bound and greatest lower bound

Orders and Groups

Definition

Let G be a group that is also a poset with partial order $\preceq$. Then G is a *partially ordered group* if whenever $g \preceq h$ and $x, y \in G$, then $xgy \preceq xhy$. This property is called *translation-invariant*. We call G a *po-group*.

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- Similarly, a po-group whose partial order is a lattice-order is an $\mathcal{L}$-group
- If the order is total then G is an *ordered group*

Examples

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The additive groups of $\mathbb{Z}, \mathbb{R}$, and $\mathbb{Q}$ are all ordered groups under the usual ordering of less than or equal to.

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Let $\mathbf{V}$ be a vector space over the rationals, with basis $\{\mathbf{b}_i : i \in I\}$. Let $\mathbf{v}, \mathbf{w} \in \mathbf{V}$, with $\mathbf{v} = \sum_{i \in I} p_i \mathbf{b}_i$ and $\mathbf{w} = \sum_{i \in I} q_i \mathbf{b}_i$. Define $\mathbf{v} \preceq \mathbf{w}$ if and only if $q_i \leq r_i$ for all $i \in I$. Then $\mathbf{V}$ is a $\mathcal{L}$-group.

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Let G be any group. Then G is *trivially ordered* if we define the order $\preceq$ by $g \preceq h$ if and only if $g = h$. With this order, then G is a partially ordered group.

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Example

Every subgroup H of a partially ordered group G is a partially ordered group itself, where H inherits the partial order from G. The same is true for subgroups of ordered groups. Note that a subgroup of a $\mathcal{L}$-group is not necessarily a $\mathcal{L}$-group.

Po-Groups

Proposition

Let G be a po-group. Then $g \preceq h$ if and only if $h^{-1} \preceq g^{-1}$

Proof.

If $g \preceq h$, then $h^{-1}gg^{-1} \preceq h^{-1}hg^{-1}$, since G is a po-group. □

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Proposition

Let G be a po-group and $g, h \in G$. If $g \vee h$ exists, then so does $g^{-1} \wedge h^{-1}$. Furthermore, $g^{-1} \wedge h^{-1} = (g \vee h)^{-1}$

Proof.

Since $g \preceq g \vee h$, it follows that $(g \vee h)^{-1} \preceq g^{-1}$. Similarly, $(g \vee h)^{-1} \preceq h^{-1}$. If $f \preceq g^{-1}, h^{-1}$, then $g, h \preceq f^{-1}$. Then $g, h \preceq f^{-1}$, and so $g \vee h \preceq f^{-1}$. Therefore, $f \preceq (g \vee h)^{-1}$. Then by definition, $g^{-1} \wedge h^{-1} = (g \vee h)^{-1}$. □

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- Using duality, we could state and prove a similar result by interchanging $\vee$ and $\wedge$

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- The elements of P are the *positive elements* of G

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- The elements of P are the *positive elements* of G
- The set $P^{-1} = G^-$ is called the *negative cone* of G
- Positive cones determine everything about the order properties of a po-group

Po-Groups

- In any group G, the existence of a positive cone determines an order on G ($g \preceq h$ if $hg^{-1} \in P$)

Proposition

A group G can be partially ordered if and only if there is a subset P of G such that:

1. $PP \subseteq P$
2. $P \cap P^{-1} = e$
3. *If $p \in P$, then $gpg^{-1} \in P$ for all $g \in G$.*

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- If, additionally, $P \cup P^{-1}$, then G can be totally ordered

$\mathcal{L}$-groups

- The lattice is always distributive in an $\mathcal{L}$-group

Theorem

If G is an $\mathcal{L}$-group, then the lattice of G is distributive. In other words, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ *and* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$*, for all* $a, b, c, \in G$.

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- Note that any lattice that satisfies the implication

$$\text{If } a \wedge b = a \wedge c \text{ and } a \vee b = a \vee c \text{ imply } b = c$$

is distributive

$\mathcal{L}$-groups

Definition

For an $\mathcal{L}$-group G, and for $g \in G$:

1. The *positive part* of g, g_+, is $g \vee e$.
2. The *negative part* of g, g_-, is $g^{-1} \vee e$.
3. The *absolute value* of g, $|g|$, is $g_+ g_-$.

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Proof.

$gg_- = g(g^{-1} \vee e) = e \vee g = g_+$. So, $g = g_+(g_-)^{-1}$. □

$\mathcal{L}$-groups

- We have the Triangle Inequality with $\mathcal{L}$-groups

Theorem (The Triangle Inequality)

Let G be an $\mathcal{L}$-group. Then for all $g, h \in G$, $|gh| \preceq |g||h||g|$.

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Theorem (The Triangle Inequality)

Let G be an $\mathcal{L}$-group. Then for all $g, h \in G$, $|gh| \preceq |g||h||g|$.

- If we require that the elements of G commute, then we recover the more traditional Triangle Inequality with two terms

$\mathcal{L}$-groups

- We can characterize abelian $\mathcal{L}$-groups using a modified Triangle Inequality

Theorem

Let G be an $\mathcal{L}$-group. Then G is abelian if and only if for all pairs of elements $g, h \in G$, $|gh| \preceq |g||h|$.

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Let G be an $\mathcal{L}$-group. Then G is abelian if and only if for all pairs of elements $g, h \in G$, $|gh| \preceq |g||h|$.

- This result comes from showing the the positive cone, G^+, is abelian

Permutations, Homomorphisms, Isomorphisms

- If G and H are po-sets and $f : G \to H$ is a function then if whenever $g_1 \preceq g_2$ for $g_1, g_2 \in G$, then $f(g_1) \preceq f(g_2)$ in H, then f is *order preserving*
- f is called an *ordermorphism*.

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- f is called an *ordermorphism*.
- If G and H are lattices then f is a *lattice homomorphism* if for all $g_1, g_2 \in G$, $f(g_1 \vee g_2) = f(g_1) \vee f(g_2)$, and $f(g_1 \wedge g_2) = f(g_1) \wedge (g_2)$
- If f is additionally bijective then f is a *lattice isomorphism*

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- If f is additionally bijective then f is a *lattice isomorphism*
- If f is a lattice homomorphism then it is also an ordermorphism
- If f is a lattice isomorphism then f^{-1} is a also a lattice isomorphism
- The set of all lattice automorphisms of a lattice G forms a group under composition of functions

Permutations, Homomorphisms, Isomorphisms

- If G and H are $\mathcal{L}$-groups and σ is both a lattice homomorphism and a group homomorphism, then σ is an *$\mathcal{L}$-homomorphism*
- The three Isomorphism Theorems translate nicely for $\mathcal{L}$-homomorphisms

Subgroups

- A *sublattice* of a lattice L is a subset S such that S is also a lattice with the ordering inherited from L
- A subgroup of S of an $\mathcal{L}$-group G is an $\mathcal{L}$*-subgroup* if S is also a sublattice of G

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- A subgroup of S of an $\mathcal{L}$-group G is an *$\mathcal{L}$-subgroup* if S is also a sublattice of G
- If A is an $\mathcal{L}$-subgroup of B which is an $\mathcal{L}$-subgroup of G which is an $\mathcal{L}$-group, then A is an $\mathcal{L}$-subgroup of G
- The intersection of $\mathcal{L}$-subgroups is again an $\mathcal{L}$-subgroup
- The kernel of an $\mathcal{L}$-homomorphism is an $\mathcal{L}$-subgroup

Subgroups

- With orders on a group, we can describe different subgroups

Definition

A subset S of a po-group G is *convex* if whenever $s, t \in S$ and $s \preceq g \preceq t$ in G, then $g \in S$.

Subgroups

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- This gives rise to *convex subgroups* of po-groups and *convex* $\mathcal{L}$-subgroups of $\mathcal{L}$-groups

Coset Orderings

- With convex subgroups we can define coset orderings

Definition

Let G be a po-group with partial order $\leq$ and S a convex subgroup of G. Let $\mathcal{R}(S)$ be the set of right cosets of S in G. On $\mathcal{R}(S)$, define $Sx \preceq Sy$ if there exists an $s \in S$ such that $sy \leq x$, for $x, y \in G$. Then $\preceq$ is a partial ordering on $\mathcal{R}(S)$, and it is called the *coset ordering* of $\mathcal{R}(S)$.

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- An entirely similar definition may be made for left cosets

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- An entirely similar definition may be made for left cosets

Theorem

Let G be a $\mathcal{L}$-group. Then a subgroup S of G is a convex $\mathcal{L}$-subgroup if and only if $\mathcal{R}(S)$ is a distributive lattice under the coset ordering.

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