Explorations of the Rubik's Cube Group

Zeb Howell

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What's the Deal with Rubik's Cubes?

- One Cube made up of twenty six subcubes called "cubelets".
- Each cubelet has one, two, or three "facelets".
- Three kinds of cubelet, defined by their number of facelets:
    1. Six cubelets with one facelet: Center cubelets
    2. Twelve cubelets with two facelets: Edge cubelets
    3. Eight cubelets with three facelets: Corner cubelets
- $12! \times 8! \times 3^8 \times 2^{12}$ combinations.
- Not all these combinations can be reached!
    - (Call this the Illegal Cube Group)

The Cube Group

Let the Cube Group G be the subgroup of S_{48} generated by:

R = (3,38,43,19)(5,36,45,21)(8,33,48,24)(25,27,32,30)(26,29,31,28)
L = (1,17,41,40)(4,20,44,37)(6,22,46,35)(9,11,16,14)(10,13,15,12)
D = (14,22,30,38)(15,23,31,39)(16,24,32,40)(41,43,48,46)(42,45,47,44)
F = (6,25,43,16)(7,28,42,13)(8,30,41,11)(17,19,24,22)(18,21,23,20)
U = (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)
B = (1,14,48,27)(2,12,47,29)(3,9,46,32)(33,35,40,38)(34,37,39,36)

Only even permutations!

Edges and Corners

Consider the set of cubelets C, and let the Cube Group act on C.

- Two orbits, $C_{corners}$ and C_{edges}.
- Let P be the group induced by the action of G on C. Then:
  1. P is the combination of all edge permutations and corner permutations.
  2. P is a subset of $(S_8 \times S_{12}) \cap A_{20}$
  3. P contains $A_8 \times A_{12}$
  4. P has order $\frac{1}{2} \times 8! \times 12!$

Orientations and Positions

- Each corner cubelet can be rotated by $\frac{2\pi k}{3}$ radians, for any integer k.
  - Equivalent to $\mathbb{Z}_3$!
- 8 corners means a direct product of $\mathbb{Z}_3$ with itself 8 times.
- Similarly, rotate each edge cubelet by $n\pi$ for any integer n to get $\mathbb{Z}_2$
- 12 edges means a direct product of $\mathbb{Z}_2$ with itself 12 times.

Time To Talk about Semi-Direct Products

Definition

Suppose that H_1 and H_2 are both subgroups of a group G. We say that G is the **semi-direct product** of H_1 by H_2, written as $H_1 \rtimes H_2$ if

- $G = H_1 \times H_2$
- H_1 and H_2 only have the identity of G in common
- H_1 is normal in G

Time To Talk About Wreath Products

Definition

Let X be a finite set where $|X| = m$, G be a group, and H a permutation group acting on X. Let G^m be the direct product of G with itself m times, and let H act on G^m by permuting the copies of G. Then the **Wreath Product** of G and H, written $G \wr H$, is defined as $G^m \rtimes H$.

Back to the Cube Group

- $C_{corners}$ acts on the set of the corner cubelets as S_8.
- The orientations of all of the corner cubelets can be described as a direct product of $\mathbb{Z}_3$ with itself eight times.
- $|S_8| = 8$
- $C_{corners}$ is the direct product of the corner orientations and the corner positions.
- $\mathbb{Z}_3^8$ is normal in $C_{corners}$
- Thus, $C_{corners} \cong (S_8 \wr \mathbb{Z}_3)$

Back to the Cube Group (continued)

- Similarly, $C_{edges} \cong (S_{12} \wr \mathbb{Z}_2)$
- We know that C_{edges} and $C_{corners}$ are separate orbits of the Cube group, so the Cube Group G $\cong C_{edges} \times C_{corners}$
- Which implies...
- The Cube Group $G \cong (\mathbb{Z}_3 \wr S_8) \times (\mathbb{Z}_2 \wr S_{12})$!

Other Fun Facts

- The order of $(\mathbb{Z}_3 \wr S_8) \times (\mathbb{Z}_2 \wr S_{12})$ is $\frac{1}{2} \cdot 8! \cdot 3^7 \cdot 12! \cdot 2^{11}$
- 43,252,003,274,489,856,000 is a big number
- That's one twelfth the order of the Illegal Cube Group
- Twelve unique orbits
- Fun Subgroups:
  1. The Slice Subgroup
  2. The Square Subgroup
  3. The Antislice Subgroup