Math 290
Exam 3
Chapter M

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Compute the inverse of the matrix A. The only Sage command you may use to justfy your answer is `.rref()` and you should clearly indicate the input and output for this command so it is clear you have not used other Sage commands to arrive at the inverse. (15 points)

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 \\ -1 & 0 & 4 & -3 \\ 0 & -1 & -2 & 4 \\ 1 & 2 & -1 & -2 \end{bmatrix}$$

$$[A \mid I_4] \xrightarrow[\text{RREF}]{} \left[I_4 \,\middle|\, \begin{matrix} 9 & 13 & 10 & 5 \\ -2 & -2 & -1 & 0 \\ 3 & 5 & 4 & 2 \\ 1 & 2 & 2 & 1 \end{matrix} \right]$$

A^{-1} by Theorems CINM, OSIS

Answer:

2. Use your answer to the previous question in a demonstration of how to use the inverse to solve the following system of equations. No credit will be given for answers obtained by other methods. (15 points)

$$\begin{aligned} x_1 - 3x_3 + x_4 &= -4 \\ -x_1 + 4x_3 - 3x_4 &= 4 \\ -x_2 - 2x_3 + 4x_4 &= 1 \\ x_1 + 2x_2 - x_3 - 2x_4 &= -5 \end{aligned}$$

SLEMM-ified

$$A\mathbf{x} = \begin{bmatrix} -4 \\ 4 \\ 1 \\ -5 \end{bmatrix},$$

A is nonsingular by (1) & Theorem NI

By Theorem SNCM

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

3. Consider the matrix B below. (40 points)

$$B = \begin{bmatrix} 1 & -1 & 7 & 3 & -3 & 6 \\ 1 & 0 & 4 & 0 & -2 & -1 \\ 1 & -1 & 7 & 4 & -4 & 8 \\ -2 & 0 & -8 & -3 & 7 & -4 \\ -1 & 1 & -7 & -4 & 4 & -8 \end{bmatrix} \xrightarrow{\text{EEF}} \left[\begin{array}{cccccc|ccccc} 1 & 0 & 4 & 0 & -2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & -2 & -1 & 0 & -5/3 & 0 & -4/3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 & 0 & -2/3 & 0 & -1/3 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2/3 & 0 & -1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$\swarrow C$ $\quad$ $\leftarrow L$

(a) Find a set P so that (i) the column space of B equals the span of P, $\mathcal{C}(B) = \langle P \rangle$, (ii) P is linearly independent, and (iii) each vector in P is a column of B.

Theorem BCS

$D = \{1, 2, 4\}$, so use columns w/ these indices

$$P = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \\ -3 \\ -4 \end{bmatrix} \right\}$$

Answer:

(b) Find a set Q so that (i) the column space of B equals the span of Q, $\mathcal{C}(B) = \langle Q \rangle$, (ii) Q is linearly independent, and (iii) Q is determined using the L matrix of extended echelon form.

$$L = \begin{bmatrix} 1 & -2/3 & 0 & -1/3 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad \mathcal{C}(B) = \mathcal{N}(L), \text{ so}$$

$$Q = \left\{ \begin{bmatrix} \\ 1 \\ \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ 0 \\ \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ 0 \\ \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2/3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Answer:

(c) Find a set R so that (i) the column space of B equals the span of R, $\mathcal{C}(B) = \langle R \rangle$, (ii) R is linearly independent, (iii) the linear independence is obvious on sight, and (iv) R is determined by a method entirely different from the previous two parts of this problem.

$\mathcal{C}(B) = \mathcal{R}(B^t)$, so row-reduce B^t and keep non zero rows as column vectors.

$$B^t \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer:

$$R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ -1 \end{bmatrix} \right\}$$

(d) Find a set T so that (i) the row space of B equals the span of T, $\mathcal{R}(B) = \langle T \rangle$, (ii) T is linearly independent, and (iii) the linear independence is obvious on sight.

Simply the rows of C above by Theorem BRS.

Answer:

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Linear independence is "obvious" in entries 1, 2 & 4

4. Suppose that A is an $m \times n$ matrix and $\mathcal{O}$ indicates a zero matrix with the subscript indicating the size. Prove that $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$. (15 points)

For $1 \le i \le m$, $1 \le j \le p$,

$$[A\mathcal{O}]_{ij} = \sum_{k=1}^{n} [A]_{ik} [\mathcal{O}]_{kj} \qquad \text{Theorem EMP}$$

$$= \sum_{k=1}^{n} [A]_{ik}\, 0 \qquad \text{Definition ZM}$$

$$= \sum_{k=1}^{n} 0$$

$$= 0$$

$$= [\mathcal{O}]_{ij}$$

So by Definition ME, $A\mathcal{O} = \mathcal{O}$

5. Suppose that M is an $m \times n$ matrix. Prove that $-M = (-1)M$. (15 points)

Add M to $(-1)M$. What is the result?

$$[M + (-1)M]_{ij} = [M]_{ij} + [(-1)M]_{ij} \qquad \text{Definition MA}$$

$$= [M]_{ij} + (-1)[M]_{ij} \qquad \text{Definition MSM}$$

$$= 0$$

$$= [\mathcal{O}]_{ij}$$

So by Definition ME, $M + (-1)M = \mathcal{O}$.

Similarly, $(-1)M + M = \mathcal{O}$.

Thus, by Property AIM, $(-1)M = -M$.