Math 290 Name: Key Dr. Beezer
Exam 4 Spring 2017
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. $P_2$ is the vector space of polynomials with degree at most 2. Find a basis for the subspace $W$. (You may assume that $W$ is a subspace.) (15 points)
$W = \left\{a + bx + cx^2 \middle| a + 3b = 0,\ 2a + b - 5c = 0\right\} \subseteq P_2$

The system $\begin{aligned} a+3b &= 0 \\ 2a+b-5c &= 0 \end{aligned}$ is equivalent to $\begin{aligned} a \quad -3c &= 0 \\ b+c &= 0 \end{aligned}$

$W = \{a+bx+cx^2 \mid a=3c,\ b=-c\} = \{3c - cx + cx^2 \mid c \in \mathbb{C}\}$

$= \{c(3-x+x^2) \mid c \in \mathbb{C}\} = \langle\{3-x+x^2\}\rangle$

It is routine to see that $B = \{3-x+x^2\}$ is also a linearly independent set, and hence a basis.

2. The set $B = \{(2, 1),\ (3, 4)\}$ is a basis for the crazy vector space, $C$. (You may assume this.) Express the vector $(4, 0)$ as a linear combination of the basis vectors. (The operations of $C$ are reproduced below.) (15 points)
$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1,\ x_2 + y_2 + 1) \qquad \alpha(x_1, x_2) = (\alpha x_1 + \alpha - 1,\ \alpha x_2 + \alpha - 1)$

Find $\alpha, \beta$ so that

$(4,0) = \alpha(2,1) + \beta(3,4)$

$= (\alpha\cdot 2 + \alpha - 1,\ \alpha\cdot 1 + \alpha - 1) + (\beta\cdot 3 + \beta - 1,\ \beta\cdot 4 + \beta - 1)$

$= (3\alpha - 1,\ 2\alpha - 1) + (4\beta - 1,\ 5\beta - 1) = (3\alpha + 4\beta - 1,\ 2\alpha + 5\beta - 1)$

Equality $\Rightarrow$ $\begin{aligned} 3\alpha + 4\beta - 1 &= 4 \\ 2\alpha + 5\beta - 1 &= 0 \end{aligned}$ $\Rightarrow$ $\begin{aligned} 3\alpha + 4\beta &= 5 \\ 2\alpha + 5\beta &= 1 \end{aligned}$ $\Rightarrow$ $\begin{aligned} \alpha &= 3 \\ \beta &= -1 \end{aligned}$

There is a solution and the solution is unique, as expected, given Theorem VRRB.

3. The following are subsets of the vector space of $2 \times 2$ matrices, $M_{22}$. Decide which are bases for $M_{22}$. Give complete justification for your answers, since a simple yes/no answer will get no credit. (25 points)

(a) $J = \left\{ \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -5 & 3 \\ -4 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 2 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -8 & 4 \end{bmatrix}, \begin{bmatrix} 7 & 3 \\ 5 & -3 \end{bmatrix} \right\}$

$\dim(M_{22}) = 2 \cdot 2 = 4$. By Theorem SSLD (or Theorem G), $J$ is "too big" to be a basis.

(b) $K = \left\{ \begin{bmatrix} 1 & -2 \\ 3 & -7 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \right\}$ The "right size", check linear independence.

$\alpha_1 \begin{bmatrix} 1 & -2 \\ 3 & -7 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ 3 & -6 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ homogeneous system w/ coefficient matrix,

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & -1 & -1 \\ 3 & 2 & 3 & 2 \\ -7 & -3 & -6 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} I_4 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

$\Rightarrow K$ linearly independent

$\Rightarrow$ w/ Theorem G, $K$ is a basis

(c) $L = \left\{ \begin{bmatrix} 1 & 0 \\ -3 & -7 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -2 & -8 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$

Also the right size.

$\alpha_1 \begin{bmatrix} 1 & 0 \\ -3 & -7 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 & 1 \\ -2 & -8 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 & -1 \\ 0 & 3 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ homo. system w/ coeff. matrix

$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ -3 & -2 & 0 & 0 \\ -7 & -8 & 3 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Any nontrivial solution provides a relation of linear dependence, so $L$ is not linearly independent, and so not a basis.

4. Suppose that $A$ is a $5 \times 7$ matrix, which is row-equivalent to a matrix in reduced row-echelon form with 3 pivot columns. (15 points)

(a) Compute the rank of $A$. $r(A) = r = 3 = \#\text{pivots}$

(b) Compute the nullity of $A$. $n(A) = \#\text{cols} - r = 7 - 3 = 4$

(c) Compute the rank of $A^t$. $r(A^t) = r(A) = 3$

(d) Compute the nullity of $A^t$. $n(A^t) = \#\text{cols} - r = 5 - 3 = 2$

5. Demonstrate the use of the three parts of Theorem TSS to show that the set $U$ is a subspace of the vector space $\mathbb{C}^2$. (15 points)

$$U = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| \, 2a + 5b = 0 \right\} \subseteq \mathbb{C}^2$$

① Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in U$? Check $2(0)+5(0)=0$, so yes, $U \neq \emptyset$.

② Consider $\begin{bmatrix} a \\ b \end{bmatrix} \in U$, $\begin{bmatrix} x \\ y \end{bmatrix} \in U$. Know $2a+5b=0$ & $2x+5y=0$.

$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+x \\ b+y \end{bmatrix}$, in $U$? Check, Yes!

$2(a+x)+5(b+y) = 2a+2x+5b+5y = (2a+5b)+(2x+5y) = 0+0=0$.

③ Consider $\alpha \in \mathbb{C}$, $\begin{bmatrix} a \\ b \end{bmatrix} \in U$. Know $2a+5b=0$.

$\alpha \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}$, in $U$? Check,

$2(\alpha a)+5(\alpha b) = \alpha(2a)+\alpha(5b) = \alpha(2a+5b) = \alpha \cdot 0 = 0$. Yes!

By Theorem TSS, $U$ is a subspace of $\mathbb{C}^2$.

6. Suppose that $V$ is a vector space, $\mathbf{v} \in V$, and $\alpha \in \mathbb{C}$. Prove that if $\alpha\mathbf{v} = \mathbf{0}$, then $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$. (Your proof should be more than simply quoting a result from the book.) (15 points)

Case 1 $\alpha = 0$. Done.

Case 2 $\alpha \neq 0$. Then

$$\begin{aligned}
\mathbf{v} &= 1\mathbf{v} && \text{Vector Space Property} \\
&= \left(\tfrac{1}{\alpha}\alpha\right)\mathbf{v} && \text{Since } \alpha \neq 0. \\
&= \tfrac{1}{\alpha}(\alpha\mathbf{v}) && \text{Associativity} \\
&= \tfrac{1}{\alpha}\mathbf{0} && \text{Hypothesis} \\
&= \mathbf{0} && \text{Theorem}
\end{aligned}$$