Math 290 Name: Key Dr. Beezer
Exam 6 Spring 2017
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. The function $T$ below is a linear transformation (you may assume this). Use a well-defined procedure to compute a matrix $A$ so that $T(\mathbf{x}) = A\mathbf{x}$. No credit will be given for an answer that does not demonstrate the use of a theorem or definition that provides a "recipe" for determining this matrix. (10 points)

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \qquad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a - 3b + 4c \\ a - b + 8c \end{bmatrix}$$

Using theorem MLTCV evaluate T with the standard unit vectors:

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

& pack into a 2x3 matrix

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -1 & 8 \end{bmatrix}$$

(or decompose $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$ into a linear combination)

2. The function $S$ below is an invertible linear transformation (you may assume this). Use a well-defined procedure to compute the inverse linear transformation, $S^{-1}$. ($P_1$ is the vector space of polynomials with degree 1 or less, and $M_{1,2}$ is the vector space of $1 \times 2$ matrices.) (15 points)

$$S\colon P_1 \to M_{1,2}, \qquad S\,(a + bx) = \begin{bmatrix} 2a + 5b & a + 3b \end{bmatrix}$$

Find preimages of a basis for the codomain; $C = \{[1\ 0], [0\ 1]\}$

$S^{-1}([1\ 0])$? $[2a+5b \quad a+3b] = [1\ 0]$

$\Rightarrow a=3,\ b=-1$

$= \{3-x\}$ so $S^{-1}([1\ 0]) = 3-x$

$S^{-1}([0\ 1])$? $[2a+5b \quad a+3b] = [0\ 1]$

$\Rightarrow a=-5,\ b=2$

$= \{-5+2x\}$ so $S^{-1}([1\ 0]) = -5+2x$

Then $S^{-1}([c\ d]) = S^{-1}(c[1\ 0] + d[0\ 1])$

$= cS^{-1}([1\ 0]) + dS^{-1}([0\ 1]) = c(3-x) + d(-5+2x)$

$= (3c-5d) + (-c+2d)x$

3. The function $T$ below is a linear transformation (you may assume this). (45 points)

$$T\colon \mathbb{C}^3 \to \mathbb{C}^4, \qquad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b-c \\ -5a-4b+c \\ -2a-2b+3c \\ -a-c \end{bmatrix}$$

(a) Compute the preimage of $\begin{bmatrix} -1 \\ -1 \\ 3 \\ -3 \end{bmatrix}$, $T^{-1}\left(\begin{bmatrix} -1 \\ -1 \\ 3 \\ -3 \end{bmatrix}\right)$.

Solve $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -3 \end{bmatrix}$

$$\begin{bmatrix} a+b-c \\ -5a-4b+c \\ -2a-2b+3c \\ -a-c \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -3 \end{bmatrix}$$

Augmented matrix of system $\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} a=2 \\ b=-2 \\ c=1 \end{matrix}$

preimage $= \left\{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$

(b) Compute the kernel of $T$, $\mathcal{K}(T)$.

$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to$ Coefficient matrix of homogeneous system

$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow a=b=c=0$ only $\Rightarrow K(T) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} = \{\underset{\sim}{0}\}$

(c) Compute the range of $T$, $\mathcal{R}(T)$.

Theorem SSRLT $\Rightarrow R(T) = \langle\{T(\underset{\sim}{e_1}), T(\underset{\sim}{e_2}), T(\underset{\sim}{e_3})\}\rangle = \left\langle\left\{\begin{bmatrix} 1 \\ -5 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \\ -1 \end{bmatrix}\right\}\right\rangle$

make these the rows of a matrix, row reduce, keep non zero rows $\Rightarrow R(T) = \left\langle\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}\right\}\right\rangle$

(d) Is $T$ injective? Explain why.

Yes, by Theorem KILT, $K(T) = \{\underset{\sim}{0}\} \Rightarrow T$ injective.

(e) Is $T$ surjective? Explain why.

$\dim(R(T)) = 3 \neq 4 = \dim(\mathbb{C}^4)$. So $R(T) \neq \mathbb{C}^4$, and by Theorem RSLT, $T$ is not surjective

↑ see (c)

(f) Is $T$ invertible? Explain why.

No, by Theorem ILTIS, since $T$ is not surjective

(g) State a simple, but fundamental theorem about the rank and nullity of any linear transformation. Determine all of the relevant quantities for $T$, with justification, and then verify the conclusion of the theorem.

$$r(T) + n(T) = \dim(\text{domain})$$
$$3 + 0 = \dim(\mathbb{C}^3) = 3$$

4. Illustrate the defining conditions of a linear transformation by proving that $S$ below is a linear transformation. ($P_1$ is the vector space of polynomials with degree 1 or less.) (15 points)

$$S\colon \mathbb{C}^2 \to P_1, \qquad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (2a-b) + (3a+2b)x$$

① $T\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+a_2 \\ b_1+b_2 \end{bmatrix}\right) = 2(a_1+a_2) - (b_1+b_2) + (3(a_1+a_2)+2(b_1+b_2))x$

$= (2a_1-b_1) + (2a_2-b_2) + ((3a_1+2b_1)+(3a_2+2b_2))x$

$= [(2a_1-b_1)+(3a_1+2b_1)x] + [(2a_2-b_2)+(3a_2+2b_2)x]$

$= T\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right)$

② $T\left(\alpha\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}\right) = (2(\alpha a) - (\alpha b)) + (3(\alpha a) + 2(\alpha b))x$

$= \alpha(2a-b) + \alpha(3a+2b)x = \alpha((2a-b)+(3a+2b)x)$

$= \alpha T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

5. Suppose that $T\colon U \to V$ is a linear transformation. We will say that vectors $\mathbf{x}, \mathbf{y} \in U$ are "related" if $\mathbf{x} - \mathbf{y} \in \mathcal{K}(T)$. Notation for this relation is $\mathbf{x} \sim \mathbf{y}$. In other words, $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{x} - \mathbf{y} \in \mathcal{K}(T)$. Prove the three conditions that establish that $T$ is an equivalence relation. (15 points)

(a) $\mathbf{x} \sim \mathbf{x}$ for all $\mathbf{x} \in U$.

$\mathbf{x} - \mathbf{x} = \mathbf{0} \in K(T)$

← since $K(T)$ is a subspace, or $T(\mathbf{0}) = \mathbf{0}$.

(b) If $\mathbf{x} \sim \mathbf{y}$, then $\mathbf{y} \sim \mathbf{x}$. Assume $\mathbf{x} - \mathbf{y} \in N(T)$. Then

$T(\mathbf{y} - \mathbf{x}) = T((-1)(\mathbf{x}-\mathbf{y})) = (-1)\,T(\mathbf{x}-\mathbf{y}) = (-1)\,\mathbf{0} = \mathbf{0}$

So $\mathbf{y} - \mathbf{x} \in N(T)$ and $\mathbf{y} \sim \mathbf{x}$.

(c) If $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \sim \mathbf{z}$, then $\mathbf{x} \sim \mathbf{z}$.

Know $T(\mathbf{x}-\mathbf{y}) = 0$ & $T(\mathbf{y}-\mathbf{z}) = \mathbf{0}$. Then

$T(\mathbf{x}-\mathbf{z}) = T((\mathbf{x}-\mathbf{y}) + (\mathbf{y}-\mathbf{z}))$

$= T(\mathbf{x}-\mathbf{y}) + T(\mathbf{y}-\mathbf{z}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$

(This proof can be generalized by replacing $K(T)$ by <u>any</u> subspace of $U$.)