Math 290 Name: Key Dr. Beezer
Exam 7 Spring 2017
Chapter R

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate vectors and matrices, row-reduce matrices, compute matrix inverses, compute determinants, and compute eigenvalues and eigenspaces. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. The function T below is a linear transformation (you may assume this). (20 points)

$$T\colon \mathbb{C}^2 \to \mathbb{C}^3, \qquad T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{bmatrix}a-b\\a+b\\a\end{bmatrix}$$

(a) For $\mathbf{v} = \begin{bmatrix}3\\1\end{bmatrix} \in \mathbb{C}^2$, compute $T(\mathbf{v})$.

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}3-1\\3+1\\3\end{bmatrix} = \begin{bmatrix}2\\4\\3\end{bmatrix}$$

(b) Compute the vector representation of $\mathbf{v}$, $\rho_B(\mathbf{v})$, relative to the basis $B = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}7\\3\end{bmatrix}\right\}$.

$$\begin{bmatrix}3\\1\end{bmatrix} = \alpha\begin{bmatrix}2\\1\end{bmatrix} + \beta\begin{bmatrix}7\\3\end{bmatrix}$$

system $\Rightarrow \begin{bmatrix}2 & 7 & 3\\1 & 3 & 1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1 & 0 & -2\\0 & 1 & 1\end{bmatrix} \Rightarrow \begin{matrix}\alpha=-2\\ \beta=1\end{matrix}$ (unique solution)

$$\rho_B\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}-2\\1\end{bmatrix}$$

(c) Compute the matrix representation of T, $M^T_{B,C}$, relative to the bases B (above) and C (below).

$$C = \left\{\begin{bmatrix}1\\-4\\2\end{bmatrix}, \begin{bmatrix}-1\\5\\-2\end{bmatrix}, \begin{bmatrix}2\\-5\\5\end{bmatrix}\right\} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$$

$$\rho_C\left(T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)\right) = \rho_C\left(\begin{bmatrix}1\\3\\2\end{bmatrix}\right) = \rho_C\left(8\begin{bmatrix}1\\-4\\2\end{bmatrix} + 7\begin{bmatrix}-1\\5\\-2\end{bmatrix} + 0\begin{bmatrix}2\\-5\\5\end{bmatrix}\right) = \begin{bmatrix}8\\7\\0\end{bmatrix}$$

$$\rho_C\left(T\left(\begin{bmatrix}7\\3\end{bmatrix}\right)\right) = \rho_C\left(\begin{bmatrix}4\\10\\7\end{bmatrix}\right) = \rho_C\left(35\begin{bmatrix}1\\-4\\2\end{bmatrix} + 29\begin{bmatrix}-1\\5\\-2\end{bmatrix} + (-1)\begin{bmatrix}2\\-5\\5\end{bmatrix}\right) = \begin{bmatrix}35\\29\\-1\end{bmatrix}$$

$$\Rightarrow M^T_{B,C} = \begin{bmatrix}8 & 35\\7 & 29\\0 & -1\end{bmatrix}$$

(d) Duplicate the computation of $T(\mathbf{v})$ from part (a), but illustrate a non-trivial use of the Fundamental Theorem of Matrix Representation with the representations from parts (b) and (c). You should, of course, get the same answer as part (a), but that is not the real purpose of the question.

$$T(\mathbf{v}) = \rho_C^{-1}\left(M^T_{B,C}\,\rho_B(\mathbf{v})\right) = \rho_C^{-1}\left(\begin{bmatrix}8 & 35\\7 & 29\\0 & -1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}\right) = \rho_C^{-1}\left(\begin{bmatrix}19\\15\\-1\end{bmatrix}\right)$$

$$= 19\begin{bmatrix}1\\-4\\2\end{bmatrix} + 15\begin{bmatrix}-1\\5\\-2\end{bmatrix} + (-1)\begin{bmatrix}2\\-5\\5\end{bmatrix} = \begin{bmatrix}2\\4\\3\end{bmatrix}$$

2. The function S below is a linear transformation (you may assume this). (P_2 is the vector space of polynomials with degree 2 or less, and $M_{1,2}$ is the vector space of 1×2 matrices.) (40 points)

$$S \colon P_2 \to M_{1,2} \qquad S\left(a + bx + cx^2\right) = \begin{bmatrix} 2a + b + 2c & -a + 3b + c \end{bmatrix}$$

(a) Choose a "nice" basis B for P_2, a "nice" basis C for $M_{1,2}$, and then compute the matrix representation of S relative to B and C, $M^S_{B,C}$.

$B = \{1, x, x^2\}$

$C = \{[1\ 0], [0\ 1]\}$

$$M^S_{B,C} = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & 1 \end{bmatrix} \quad \text{"on sight"}$$

(b) Using the bases D and E below, of P_2 and $M_{1,2}$ respectively, use the definition of a matrix representation to compute the matrix representation of S relative to D and E, $M^S_{D,E}$.

$D = \{1 + x + 4x^2,\ 1 - x - 3x^2,\ -x - 4x^2\}$ $\qquad$ $E = \{[-3\ \ 4],\ [2\ \ -3]\}$

$$\rho_E(S(1+x+4x^2)) = \rho_E([11\ 6]) = \rho_E((-45)[-3\ 4] + (-62)[2\ -3]) = \begin{bmatrix} -45 \\ -62 \end{bmatrix}$$

$$\rho_E(S(1-x-3x^2)) = \rho_E([-5\ -7]) = \rho_E(29[-3\ 4] + 41[2\ -3]) = \begin{bmatrix} 29 \\ 41 \end{bmatrix}$$

$$\rho_E(S(-x-4x^2)) = \rho_E([-9\ -7]) = \rho_E(41[-3\ 4] + 57[2\ -3]) = \begin{bmatrix} 41 \\ 57 \end{bmatrix}$$

$$M^S_{D,E} = \begin{bmatrix} -45 & 29 & 41 \\ -62 & 41 & 57 \end{bmatrix}$$

(c) Use change of basis matrices and your representation in part (a) to re-compute the representation in part (b). Show the relevant product of matrices and compute the necessary products to duplicate the earlier result.

$$M^S_{D,E} = C_{C,E}\, M^S_{B,C}\, C_{D,B}$$

$$C_{D,B} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 4 & -3 & -4 \end{bmatrix} \qquad C_{C,E} = C_{E,C}^{-1} = \begin{bmatrix} -3 & 2 \\ 4 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & -2 \\ -4 & -3 \end{bmatrix}$$

on sight $\qquad$ on sight

$$= \begin{bmatrix} -3 & -2 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 4 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -45 & 29 & 41 \\ -62 & 41 & 57 \end{bmatrix}$$

3. The function S below is an invertible linear transformation (you may assume this). Use a matrix representation to compute the inverse linear transformation, S^{-1}. No credit will be given for a method that does not make significant use of the matrix representation. (P_1 is the vector space of polynomials with degree 1 or less, and $M_{1,2}$ is the vector space of 1×2 matrices.) (20 points)

$S\colon P_1 \to M_{1,2} \qquad S\left(a + bx\right) = \begin{bmatrix} 2a + 5b & a + 3b \end{bmatrix}$

$B = \{1, x\}$ $\quad C = \{[1\ 0], [0\ 1]\}$ "nice" bases

$M^{S}_{B,C} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ "on sight" then $M^{S^{-1}}_{C,B} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$S^{-1}([x\ y]) = \rho_B^{-1}\left(M^{S^{-1}}_{C,B}\, \rho_C([x\ y])\right) = \rho_B^{-1}\left(\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)$

Oops! not the "x" of the polynomials

$= \rho_B^{-1}\left(\begin{bmatrix} 3x - 5y \\ -x + 2y \end{bmatrix}\right) = (3x - 5y)(1) + (-x + 2y)\, x$

Better: $S^{-1}([a\ b]) = (3a - 5b) + (-a + 2b)\, x$

4. Find a basis for P_2 that will yield a matrix representation of T that is a diagonal matrix, along with a complete explanation. (P_2 is the vector space of polynomials with degree 2 or less.) (20 points)

$T\colon P_2 \to P_2, \qquad T\left(a + bx + cx^2\right) = (5a + 6b + 12c) + (-9a - 10b - 18c)x + (3a + 3b + 5c)x^2$

$B = \{1, x, x^2\}$ "standard basis"

$M^{T}_{B,B} = \begin{bmatrix} 5 & 6 & 12 \\ -9 & -10 & -18 \\ 3 & 3 & 5 \end{bmatrix}$ Find a basis of eigenvectors using Sage

$\underline{\lambda = 2}$ $\quad \underset{\sim}{x_1} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ $\qquad \underline{\lambda = -1}$ $\quad \underset{\sim}{x_2} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ $\quad \underset{\sim}{x_3} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

$\{x_2, x_3\}$ is a linearly independent set

"Un-coordinatize", ρ_B^{-1}

$C:$ $\quad \underset{\sim}{x_1} = 2 - 3x + x^2$ $\qquad \underset{\sim}{x_2} = 2 - x^2$ $\qquad \underset{\sim}{x_3} = 2x - x^2$ ← basis of eigenvectors of T

Not requested:

$M^{T}_{C,C} = \begin{bmatrix} 2 & & \\ & -1 & \\ & & -1 \end{bmatrix}$ eigenvalues on diagonal.