Valuation Rings

Rachel Chaiser
May 1, 2017
University of Puget Sound

Definition: Valuation

- F - field
- G - totally ordered additive abelian group
- For all $a, b \in F$, $\nu : F \to G \cup \{\infty\}$ satisfies:
  1. $\nu(ab) = \nu(a) + \nu(b)$
  2. $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$
  3. $\nu(0) := \infty$
- If ν is surjective onto $G = \mathbb{Z}$ then ν is **discrete**

Example: p-adic Valuation

- Fix a prime $p \in \mathbb{Z}$
- Any $\frac{r}{s} \in \mathbb{Q}^*$ can be written uniquely as $\frac{r}{s} = p^k \frac{a}{b}$, $\frac{a}{b} \in \mathbb{Q}^*$, $p \nmid ab$
- Define $\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$, $\nu_p(\frac{r}{s}) = k$
- For example, 3-adic valuation:
  - $\nu_3(1) = 0$
  - $\nu_3(12) = \nu_3(3^1 \cdot 4) = 1$
  - $\nu_3(\frac{5}{9}) = \nu_3(5 \cdot 3^{-2}) = -2$

Structures

Definitions: Structures

- **Definition (Value Group)**
  The subgroup of G, $\nu(F^*) = \{\nu(a) \mid a \in F^*\}$
- **Definition (Valuation Ring)**
  The subring of F, $V = \{a \in F \mid \nu(a) \geq 0\}$
- **Definition (Discrete Valuation Ring)**
  If ν is discrete then V is a discrete valuation ring (DVR)

Example: p-adic Structures

- Recall: $\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$, $\nu_p(p^k \frac{a}{b}) = k$ where $p \nmid ab$
- The value group of ν_p is $\mathbb{Z}$
- Assume $\frac{r}{s}$ is in lowest terms
- The valuation ring of ν_p is $\mathbb{Z}_{(p)} = \{\frac{r}{s} \mid p \nmid s\}$, the **$p$-adic integers**
- The 3-adic integers:
  - $\frac{5}{9} \notin \mathbb{Z}_{(3)}$ while $1, 12 \in \mathbb{Z}_{(3)}$
  - $\mathbb{Z} \subset \mathbb{Z}_{(3)}$
  - $\frac{n}{a} \in \mathbb{Z}_{(3)}$ where $n \in \mathbb{Z}$ and $\gcd(a, 3) = 1$

Properties

Properties of Valuation Rings

For general ν:

- For all $a, b \in V$, $\nu(a) \leq \nu(b) \iff b \in \langle a \rangle$
- The ideals of V are totally ordered by set inclusion
- V has unique maximal ideal $M = \{a \in V \mid \nu(a) > 0\}$

For discrete ν:

- $t \in V$ with $\nu(t) = 1$ is a **uniformizer**
- $M = \langle t \rangle$

DVRs are Noetherian

Proof.

Let $I \neq \langle 0 \rangle$ be an ideal of V. Then for some $a \in I$ there is a least integer k such that $\nu(a) = k$. Let $b, c \in I$ and suppose $b = ac$. Then $\nu(b) = \nu(a) + \nu(c) = k + \nu(c) \geq k$. Thus I contains every $b \in V$ with $\nu(b) \geq k$, and so the only ideals of V are $I_k = \{b \in V \mid \nu(b) \geq k\}$. These ideals then form a chain $V = I_0 \supset I_1 \supset I_2 \supset \cdots \supset \langle 0 \rangle$. $\square$

DVRs are PIDs

Proof.

Let $t \in V$ be a uniformizer. For $x \in \left\langle t^k \right\rangle$,

$$\nu(x) = \nu(at^k) = \nu(a) + k\nu(t) = \nu(a) + k.$$

Thus, we can take $I_k = \left\langle t^k \right\rangle$. $\square$

Remark

This illustrates that $\nu(a) \leq \nu(b) \iff b \in \langle a \rangle$ for all $a, b \in V$

Corollary

Every nonzero ideal of V is a power of the unique maximal ideal, $\langle t \rangle$.

Example: p-adic ideals

- $M = \{\frac{r}{s} \in \mathbb{Z}_{(p)} : p \mid r\} = \langle p \rangle$
- $\mathbb{Z}_{(p)} = \left\langle p^0 \right\rangle \supset \left\langle p^1 \right\rangle \supset \left\langle p^2 \right\rangle \supset \left\langle p^3 \right\rangle \supset \cdots \supset \langle 0 \rangle$
- 3-adic ideals:
    - Maximal ideal $\langle 3 \rangle$
    - $\mathbb{Z}_{(3)} = \langle 1 \rangle \supset \langle 3 \rangle \supset \left\langle 3^2 \right\rangle \supset \left\langle 3^3 \right\rangle \supset \cdots \supset \langle 0 \rangle$

Example: Generalized p-adic Valuation

- Let D be a UFD with field of fractions K
- Fix a prime element p of D
- Any $x \in D$ can be written uniquely as $x = ap^k$ where $p \nmid a$
- Any $y \in K^*$ can be written uniquely as $y = qp^k$
  - $q \in K^*$ is the quotient of $r, s \in D$ such that $p \nmid r$, $p \nmid s$
- Define $\nu : K \to \mathbb{Z} \cup \{\infty\}$, $\nu(y) = k$

Thank you!

Questions?