Valuation Rings

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Abstract

This paper surveys the properties of valuation rings and discrete valuation rings from the perspective of a consequence of a valuation. We begin with an introductory glance at valuations before moving on to study the properties of valuation rings and discrete valuation rings. We focus primarily on proving properties of valuation rings, culminating in equivalences for both valuation rings and discrete valuation rings. The equivalences then provide motivation for describing a method of recovering the valuation associated with a valuation ring that was not constructed from a valuation.

1 Introduction

From our perspective, a valuation ring is an algebraic structure that arises from a particular type of function called a valuation. Valuations provide a way to assign sizes to the elements of a field in a way that is consistent and allows us to compare elements by their sizes. One can extend this notion to define the distance between two elements which induces a topology (for more on this see [6]).

Valuation rings find use in a variety of mathematical fields. In particular, discrete valuation rings are especially useful in number theory as they are unique factorization domains – the ability to factor a number uniquely is of great importance in number theory. Other fields in which valuation rings are applied include algebraic and analytic geometry, and complex analysis [9].

Aside from their applications in various fields of mathematics, valuation rings are intriguing to study in their own right due to their interesting structure. Our purpose here is to give an introductory survey of the properties of valuations and valuation rings that is accessible to the student familiar with undergraduate abstract algebra. We begin with a brief overview of valuations as a lead into studying the properties of valuation rings. Though we will primarily conceptualize valuation rings as an algebraic structure arising from the definition of a valuation, there are several equivalent conditions for valuation rings. Thus, our study will culminate in the process by which we can recover, up to isomorphism, the valuation associated to a particular valuation ring that did not arise from the definition of a valuation.

2 Valuations

It is often useful to assign sizes to the elements of a field to obtain some metric for measuring these field elements. In order for our measurements to be useful, we would like to be able to say definitively whether two given field elements are larger or smaller than one another, or have the same size. Thus, we need a function from a field to a totally ordered set. We will take our codomain to be a group.

Definition 2.1. Let F be a field, $F^* = F \setminus \{0\}$, and G be a totally ordered abelian group under addition. We define a **valuation** on F to be a map $\nu : F^* \to G$ satisfying:

1. $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in F$; and
2. if $a + b \neq 0$, then $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for all $a, b \in F$.

Additionally, when we wish to extend ν to all of F, we write $\nu : F \to G \cup \{\infty\}$ and use the convention $\nu(0) = \infty$. (This clearly satisfies the above conditions.) In this case, ∞ is now the largest element of G and respects the usual conventions when handling the symbol; that is, $\infty + k = \infty$ for all $k \in G$ [5, 6].

Definition 2.2. If ν is surjective onto $G = \mathbb{Z}$, then ν is a **discrete valuation** [2].

Remark. Condition 1 is equivalent to saying ν is a group homomorphism from the multiplicative group of F, F^*, to G [3].

One may wonder why we require G to be an *additive* group. In fact, this is not required. There is an equivalent definition of a valuation for the case where G is a *multiplicative* group; however, this variation occurs significantly less frequently in the literature on valuation rings. Thus, we will restrict our study to the case where G is an additive group [5].

We will maintain the notation of Definition 2.1 throughout our study of valuation rings so that ν will denote a valuation, F will denote the domain of ν, G will denote the codomain of ν, and for any ring R, $R^* = R \setminus \{0\}$. We will often use F^* when we wish to avoid complications that may arise by considering $\nu(0)$. The reader should keep in mind that we are always able to extend ν to all of F with the convention $\nu(0) = \infty$, even when we express a particular valuation as $\nu : F^* \to G$. Note that some authors choose instead to maintain the notation $\nu : F \to G \cup \{\infty\}$; however, this notation can become cumbersome for our purposes.

2.1 Properties of Valuations

As our primary view of valuation rings will be from the perspective of an algebraic structure arising from a valuation, it is integral to our study to understand some of the basic properties of valuations. The following propositions ought to build intuition for the behavior of valuations and prepare the reader for the proofs in Section 3. For more on valuation theory, we refer the reader to [4, 6].

Proposition 2.3. *For any valuation ν, $\nu(1) = 0$.*

Proof. Let $a \in F^*$. Then $\nu(1) = (\nu(1) + \nu(a)) - \nu(a) = \nu(1a) - \nu(a) = \nu(a) - \nu(a) = 0$. □

This result fits nicely with our conception of a valuation as a way to measure the size of field elements, as we do not want repeated multiplication by 1 to contribute to the valuation of a. Additionally, recall that ν is a group homomorphism from a multiplicative group to an additive group. From this view it must be the case that $\nu(1) = 0$.

Proposition 2.4. *Let $k \in \mathbb{Z}$. Then $\nu(a^k) = k\nu(a)$ for all $a \in F^*$.*

Proof. When $k \geq 0$ this follows easily from condition 1 of Definition 2.1, so we will only prove this directly for $k = -1$. Consider $\nu(a) + \nu(a^{-1}) = \nu(aa^{-1}) = \nu(1) = 0$. Thus $\nu(a^{-1}) = -\nu(a)$. □

We now provide two preliminary examples of valuations before proceeding to study the structures associated with a valuation.

Example 2.5. For any field F, we can define the **trivial valuation** $\nu : F^* \to \{0\}$, $\nu(x) = 0$, with the convention $\nu(0) = \infty$ [5].

Example 2.6. Fix $p \in \mathbb{Z}$ where p is prime. Then any element $\frac{r}{s} \in \mathbb{Q}^*$ can be written uniquely as $\frac{r}{s} = p^k \frac{a}{b}$ where $\frac{a}{b} \in \mathbb{Q}^*$ and $p \nmid ab$. This suggests a natural valuation $\nu_p : \mathbb{Q}^* \to \mathbb{Z}$ defined by $\nu_p(\frac{r}{s}) = k$, where ν_p measures the extent to which p is a factor of $\frac{r}{s}$. This valuation occurs frequently in number theory and is called ***p*-adic valuation**. p-adic valuation is a discrete valuation, which may come as no surprise given that the codomain of ν_p is $\mathbb{Z}$; however, we must show that ν_p is onto. Observe that for any $k \in \mathbb{Z}$, we simply need to produce a rational number $\frac{r}{s} \in \mathbb{Q}^*$ that contains k factors of p. This is always possible, as we can simply let $\frac{r}{s} = p^k$. Of course, there are also many other choices we could have made [6].

3 Valuation Rings

We are now prepared to begin our study of valuation rings. However, rings are not the only algebraic structure that arise from the definition of a valuation. Being a map from a field to a group, it should come as no surprise that we can define substructures of both the domain and codomain of a valuation. We begin with codomain.

Definition 3.1. The image of ν, $\nu(F^*)$, is a subgroup of G called the **value group** of ν.

The proof that $\nu(F^*)$ is a group is not difficult, but we will provide it anyway. Additionally, for the reader that is unfamiliar with valuations, the proofs in this section should give an example of how valuations will be used in later proofs.

Proof. The identity of $\nu(F^*)$ is $\nu(1) = 0$. For any $\nu(a) \in \nu(F^*)$, we have $0 = \nu(1) = \nu(aa{-}1) = \nu(a) - \nu(a)$. Thus the inverse of $\nu(a)$ is $\nu(a^{-1})$. Let $\nu(a), \nu(b) \in \nu(F^*)$. Then $\nu(a) + \nu(b) = \nu(ab) \in \nu(F^*)$ and so $\nu(F^*)$ is closed. □

Definition 3.2. The set $V = \{a \in F \mid \nu(a) \geq 0\}$ is a subring of F called the **valuation ring** of ν [5].

Definition 3.3. If ν is a discrete valuation, then the valuation ring of ν is called a **discrete valuation ring**, abbreviated DVR [2].

It is also common to refer to V as a valuation ring of F. Notice that V is unique with respect to ν but not with respect to F. Again, the proof that V is a ring is not difficult, but we will include it.

Proof. We can see that both 1 and 0 are in V as $\nu(1) = 0$ and $\nu(0) = \infty$. To see that V is closed under addition and multiplication, let $a, b \in V$ so that $\nu(a) \geq 0$ and $\nu(b) \geq 0$. Then $\nu(ab) = \nu(a) + \nu(b) \geq 0$ and $\nu(a + b) \geq \min\{\nu(a), \nu(b)\} \geq 0$. □

Proposition 3.4. *V is an integral domain.*

Proof. Let $a, b \in V$ such that $ab = 0$. Then $\infty = \nu(ab) = \nu(a) + \nu(b)$. Thus, it must be the case that either $\nu(a) = \infty$ or $\nu(b) = \infty$, and so we have either $a = 0$ or $b = 0$. □

We now return to our examples from the previous section to give examples of the structures arising from a particular valuation.

Example 3.5. For the trivial valuation $\nu : F^* \to \{0\}$, $\nu(x) = 0$, the value group of ν is $\{0\}$ and the valuation ring of ν is F.

This constitutes a proof that any field is also a valuation ring. However, this is not very interesting and we often wish to restrict our study to valuation rings that are not also fields.

Example 3.6. Recall that p-adic valuation is the map $\nu_p : \mathbb{Q}^* \to \mathbb{Z}$ defined by $\nu_p(\frac{r}{s}) = \nu_p(p^k \frac{a}{b}) = k$ where $p \nmid ab$, and that ν_p is a discrete valuation. Thus the value group of ν_p is $\mathbb{Z}$. Now, assuming $\frac{r}{s}$ is in lowest terms, the valuation ring of ν_p is $V = \{\frac{r}{s} \mid p \nmid s\} \cup \{0\}$. This follows because if $p \mid s$ then there are no factors of p in r and thus $\nu_p(\frac{r}{s}) < 0$. The elements of V are called the **p-adic integers** and are denoted $\mathbb{Z}_{(p)}$ [6].

3.1 Properties of Valuation Rings

Valuation rings have many interesting properties, and we are now prepared to begin our survey of them. However, we will not provide a comprehensive list of these properties as such a task is outside of the scope of this paper and requires knowledge of topics that we do not assume the reader to be familiar with, such as localization and integral closure. For more on these topics and their applications to the study of valuation rings, we refer the reader to [3, 5, 9].

Throughout this section we will assume that V is the valuation ring of $\nu : F^* \to G$. Recall that whenever we need 0 we simply extend ν to all of F with the convention $\nu(0) = \infty$.

Proposition 3.7. *For all $a \in F^*$, $a \in V$ or $a^{-1} \in V$ (inclusive).*

Proof. Since $\nu(a) + \nu(a^{-1}) = \nu(aa^{-1}) = \nu(1) = 0$, it follows that $\nu(a) \geq 0$ or $\nu(a^{-1}) \geq 0$. Thus, we have $a \in V$ or $a^{-1} \in V$. □

Corollary 3.8. *The group of units of V is $U = \{a \in V \mid \nu(a) = 0\}$.*

Proof. Let $a \in V$ with $a \neq 0$. If $a \in U$ then $a^{-1} \in U$ by definition, and so $a^{-1} \in V$. Thus $\nu(a^{-1}) = -\nu(a) \geq 0$, but this is only the case if $\nu(a) = -\nu(a) = 0$ [5]. □

Proposition 3.9. *For all $a, b \in V$ we have $\nu(a) \leq \nu(b)$ if and only if $b \in \langle a \rangle$.*

Proof. Clearly this is true when $a = 0$, so we will assume $a \neq 0$.
($\Rightarrow$) If $\nu(a) \leq \nu(b)$ then $\nu(ba^{-1}) = \nu(b) - \nu(a) \geq 0$, so $ba^{-1} \in V$. Thus $b = (ba^{-1})a \in \langle a \rangle$.
($\Leftarrow$) If $b \in \langle a \rangle$ then, for some $c \in V$, $b = ca$, and $\nu(b) = \nu(c) + \nu(a) \geq \nu(a)$ [5]. □

Corollary 3.10. *For all $a \in V$, $\nu(-a) = \nu(a)$.*

Proof. Clearly $-a \in \langle a \rangle$, and so $\nu(a) \leq \nu(-a)$. Similarly, $\nu(-a) \leq \nu(a)$. Thus $\nu(-a) = \nu(a)$. □

Consider what this result means under our conceptualization of a valuation as a way to measure the size of a field element. Analogous to absolute values (though valuations are not absolute values), a field element has the same size as its additive inverse. Considering our example of p-adic valuation, this result is precisely what we would expect. For any nonzero rational number r, the extent to which p is a factor of r is precisely the same as the extent to which p is a factor of $-r$.

Proposition 3.11. *The field of fractions of V is F.*

Proof. Since $V \subseteq F$, it is sufficient to show that for all $a \in F$, $a \neq 0$, there exists some $b \in V$ such that $ab \in V$. To see this, observe that if $r = ab$, then $rb^{-1} = a \in F$ and so we can express a as r/b; that is, we can express any nonzero element in F as the quotient of two elements from V. Let $a \in F$ such that $a \neq 0$. Then by Proposition 3.7, either $a \in V$ or $a^{-1} \in V$. If $a \in V$, then $aa \in V$ by closure. If $a^{-1} \in V$, then $aa^{-1} = 1 \in V$ [2]. □

Proposition 3.12. *V has a unique maximal ideal $M = \{a \in V \mid \nu(a) > 0\}$.*

Proof. Clearly $0 \in M$, so M is nonempty. Now, let $a, b \in M$ and $r \in R$. Then $\nu(a + b) \geq \min\{\nu(a), \nu(b)\} > 0$ and $\nu(ab) = \nu(a) + \nu(b) > 0$. Thus M is a subring of V. Also, $\nu(ra) = \nu(r) + \nu(a) > 0$. Thus M is an ideal of V. Now, let I be an ideal of V such that $M \subset I$. Then I contains an element with valuation 0, which is a unit by Corollary 3.8, so $I = V$. Thus M is maximal. All that is left is to show that M is unique. Notice that M is the set of all nonunits of V. Suppose we have another maximal ideal J of V. Then J must contain no units else $J = V$, so $J \subseteq M$. Since both M and J are maximal, we have $J = M$ [5]. □

Definition 3.13. Any commutative ring with a unique maximal ideal is called a **local** ring [5].

Proposition 3.14. *The ideals of V are totally ordered by set inclusion.*

Proof. Let I and J be ideals of V such that $I \not\subset J$. Let $a \in I \setminus J$ (so $a \neq 0$), and let $b \in J$. We want to show that $b \in I$. If $b = 0$ then we are done, so assume $b \neq 0$. Now, consider ba^{-1}. By Proposition 3.7, we have $ba^{-1} \in V$ or $ab^{-1} \in V$. If $ab^{-1} \in V$, then $(ab^{-1})b = a \in J$ which is a contradiction. Thus $ba^{-1} \in V$ and so $(ba^{-1})a = b \in I$ [2]. □

3.2 Properties of Discrete Valuation Rings

As discrete valuation rings are a specific case of valuation rings, their properties are a bit more restrictive than those of general valuation rings. For instance, DVRs are Euclidean domains but not fields. Although all Euclidean domains are principal ideal domains (PIDs) and all PIDs are Noetherian, we will prove each of these properties separately, as their proofs provide insight into the structure of discrete valuation rings. In particular, DVRs have a special element called a uniformizer which is guaranteed by the surjectivity of a discrete valuation. This element will play a pivotal role in the proofs of this section.

Throughout this section we will assume V is the discrete valuation ring for $\nu : F^* \to \mathbb{Z}$.

Definition 3.15. Any element $t \in V$ with $\nu(t) = 1$ is called a **uniformizer** of V [2].

Proposition 3.16. *If t is a uniformizer of V, then t is irreducible.*

Proof. Suppose t factors as $t = ab$ where $a, b \in V$. Then $1 = \nu(t) = \nu(a) + \nu(b)$. Thus, either $\nu(a) = 0$ or $\nu(b) = 0$ [1]. □

Corollary 3.17. *Let t be a uniformizer of V. Then t' is a uniformizer of V if and only if t and t' are associates [10].*

Proposition 3.18. *Let M be the unique maximal ideal of V and let $t \in V$. Then $M = \langle t \rangle$ if and only if t is a uniformizer of V.*

Proof. ($\Leftarrow$) Suppose $\nu(t) = 1$. Since M is the unique maximal ideal, $\langle t \rangle \subseteq M$. Since ν is discrete, we can write $M = \{a \in V | \nu(a) \geq 1\}$. Now, let $a \in M$ so that $\nu(a) \geq 1$. Consider $\nu(at^{-1}) = \nu(a) - \nu(t) \geq 1 - 1 = 0$, and so $at^{-1} \in V$. Consequently, $(at^{-1})t = a \in \langle t \rangle$.
($\Rightarrow$) Suppose $M = \langle t \rangle$. By surjectivity of ν, there exists some $z \in M$ such that $\nu(z) = 1$. Let $z = ct$ for some $c \in V$. Then $1 = \nu(z) = \nu(c) + \nu(t)$, so we have $\nu(t) = 1 - \nu(c) \geq 1$. If $\nu(c) \neq 0$, then $\nu(t) < 0$ which gives a contradiction. Thus $\nu(t) = 1$ [2]. □

Corollary 3.19. *V is not a field.*

Proof. The only maximal ideal of a field is the zero ideal $\langle 0 \rangle$, as all nonzero field elements are units. Let $M = \langle t \rangle$. Then $\nu(t) = 1 \neq \infty$ and so $t \neq 0$. Thus the maximal ideal of V cannot be $\langle 0 \rangle$ [10]. □

Proposition 3.20. *V is Noetherian.*

Proof. Let $I \neq \langle 0 \rangle$ be an ideal of V. Then for some $a \in I$ there is a least integer k such that $\nu(a) = k$. Now let $b, c \in I$ and suppose $b = ac$. Then $\nu(b) = \nu(a) + \nu(c) = k + \nu(c) \geq k$ because $c \in V$. Therefore, I contains every $b \in V$ such that $\nu(b) \geq k$, and so the only ideals of V are $I_k = \{b \in V \mid \nu(b) \geq k\}$. Recall that the ideals of V are totally ordered. Thus these ideals form a chain $V = I_0 \supset I_1 \supset I_2 \supset \cdots \supset \langle 0 \rangle$ and so V is Noetherian. □

Corollary 3.21. *V is a PID.*

Proof. Let $t \in V$ be a uniformizer. Then in the previous proof we can take $I_k = \langle t^k \rangle$. By Proposition 3.18, we can say then that every nonzero ideal of V is a power of M [3]. □

Proposition 3.22. *V is a Euclidean domain.*

Proof.
1. For all nonzero $a, b \in V$ we have $\nu(a) \leq \nu(ab)$ by Proposition 3.9, since $ba \in \langle a \rangle$.
2. Let $a, b \in V$ with $b \neq 0$. If $ab^{-1} \in V$ then we can write $a = (ab^{-1})b + 0$. If $ab^{-1} \notin V$ then $\nu(ab^{-1}) = \nu(a) - \nu(b) < 0$, so we have $\nu(a) < \nu(b)$. Thus we can write $a = 0b + a$ [7].

□

The properties we have explored throughout this section suggest a natural conceptualization of a discrete valuation ν. Fix a uniformizer t of the valuation ring V of ν. Then for any $x \in F^*$, we can write x uniquely as $x = at^k$ where $k = \nu(x)$ and $a \in F^*$ such that $t \nmid a$ [10].

3.3 Equivalences

Although we have conceptualized valuation rings as arising from a valuation, there are many equivalent conditions for both valuation rings and discrete valuation rings. As the following theorems illustrate, it is not unlikely that we will find ourselves with a valuation ring which we did not construct from a valuation.

Theorem 3.23. *For an integral domain R with field of fractions K, the following are equivalent.*
1. *There is a valuation ν of K for which R is the valuation ring.*
2. *For all $a \in K$, either $a \in R$ or $a^{-1} \in R$.*
3. *The ideals of R are totally ordered by set inclusion.*

Proof. The proof of this theorem can be found in [8]. The reader should note that many authors take (2) as the definition of a valuation ring and give (1) as a theorem. □

Theorem 3.24. *Let R be a Noetherian integral domain with unique maximal ideal $M \neq \langle 0 \rangle$; that is, R is not a field. The following are equivalent.*
1. *R is a DVR (under our definition).*
2. *R is a PID.*
3. *M is principal.*
4. *Every nonzero ideal is a power of M.*

Proof. The proof of this theorem can be found in [2]. Again, many authors take the definition of a discrete valuation ring to be a local PID, proving our definition as a theorem. □

Now that we have equivalent characterizations of discrete valuation rings, let us return to the natural conceptualization of a discrete valuation mentioned at the close of the previous section. If we find ourselves with a discrete valuation ring V without an associated valuation, there is then a natural valuation to define. Let K be the field of fractions of V and let $t \in V$ be a prime element. Then we can express any $x \in K^*$ uniquely as $x = at^k$ where $k \in \mathbb{Z}$ and $a \in K^*$ such that $t \nmid a$. Now we can define $\nu_t : K^* \to \mathbb{Z}$ by $\nu_t(x) = k$. This is in fact a generalization of p-adic valuation [5].

4 Recovery

As we saw in the previous section, it is entirely possible to have a valuation ring that we did not construct directly from a valuation. The following theorem guarantees that we can recover the associated valuation, up to isomorphism, and provides us with a method for doing so.

Theorem 4.1. *Let V be the valuation ring of a valuation $\nu : F^* \to G$. Then ν can be recovered completely from V up to isomorphism.*

Proof. By Corollary 3.8, $\ker \nu = U$, the group of units of V, so by the First Isomorphism Theorem we have $\nu(F^*) \cong F^*/U$. Note that both F^* and U are groups under multiplication and so F^*/U is a factor group under multiplication. Now, define $\bar{\nu} : F^* \to F^*/U$ to be the canonical homomorphism [7]. For $\bar{\nu}$ to be a valuation, F^*/U must be totally ordered. However, since F^*/U is isomorphic to $\nu(F^*)$, a subgroup of G, F^*/U inherits the total order of G. Explicitly, the total order of G is defined as, for $a, b \in F^*$, $\nu(a) \leq \nu(b)$ if and only if $ba^{-1} \in V$. This follows because $\nu(a) \leq \nu(b)$ if and only if $0 \leq \nu(b) - \nu(a) = \nu(ba^{-1})$. Thus, for $r_1U, r_2U \in F^*/U$, we have $r_1U \leq r_2U$ if and only if $r_2r_1^{-1} \in V$. Finally, for ν and $\bar{\nu}$ to be isomorphic valuations, there must be some order-preserving homomorphism ϕ of their respective value groups such that $\phi \circ \bar{\nu} = \nu$. This will be the homomorphism $\phi : F^*/U \to G$ defined by $\phi(r_1U) = \nu(r_1)$, which turns multiplication in F^*/U into addition in G. To see that this is in fact a homomorphism of groups, let $r_1U, r_2U \in F^*/U$. Then $\phi((r_1r_2)U) = \nu(r_1r_2) = \nu(r_1) + \nu(r_2) = \phi(r_1U) + \phi(r_2U)$. We must also show that ϕ is well-defined. Suppose $r_1U = r_2U$. Then for some $u \in U$, $r_1u = r_2$. Consequently, $\phi(r_1U) = \nu(r_1) = \nu(r_1) + \nu(u) = \nu(r_1u) = \nu(r_2) = \phi(r_2U)$. All that is left to show is that ϕ is order-preserving, but this follows directly from our above discussion on the ordering of F^*/U. Since $r_1U \leq r_2U$ if and only if $r_2r_1^{-1} \in V$ if and only if $\nu(r_1) \leq \nu(r_2)$, we have that ϕ is order-preserving. □

This construction can be performed without having first known the valuation $\nu : F^* \to G$ by using the field of fractions and group of units of a given valuation ring, since Proposition 3.11 tells us that the field of fractions of any valuation ring is the domain of the associated valuation. Note that we cannot recover all of G unless ν is surjective, we can only recover the value group of ν [5].

5 Conclusion

This concludes our survey of the properties of valuations rings. We began with a useful tool – a function that provides some notion of size within a field – and constructed further algebraic structures with that tool. We discovered the many interesting features of the structure of valuation rings, both general and discrete, and provided equivalent characterizations for these structures, culminating in a method for recovering the tool we began with.

References

[1] A. Altman and S. Kleiman, *A Term of Commutative Algebra*, Worldwide Center of Mathematics, 2013, available at http://web.mit.edu/18.705/www/13Ed-2up.pdf.

[2] R. Ash, "Valuation Rings." *A Course in Commutative Algebra*, 2003, available at http://www.math.uiuc.edu/~r-ash/ComAlg.html.

[3] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Westview Press, Oxford, 1969.

[4] P. Clark, "Lecture Notes on Valuation Theory," Lecture Notes, available at http://math.uga.edu/~pete/MATH8410.html.

[5] A. Gathmann, *Commutative Algebra*, Lecture Notes, 2013, available at http://www.mathematik.uni-kl.de/~gathmann/class/commalg-2013/main.pdf.

[6] F. Q. Gouvêa, *p-adic Numbers: An Introduction,* 2nd ed., Springer-Verlag, Berlin, 1997.

[7] T. Judson, *Abstract Algebra: Theory and Applications*, 2016, available at http://abstract.pugetsound.edu/aata.

[8] A. Ogus, "Valuation Rings," Lecture Notes, 2008, available at https://math.berkeley.edu/~ogus/Math\%20_256A--08/valuationrings.pdf.

[9] M. Reid, *Undergraduate Commutative Algebra*, Cambridge University Press, 1995.

[10] A. Sutherland, "Absolute Values and Discrete Valuations," Lecture Notes, 2015, available at http://math.mit.edu/classes/18.785/2015fa/LectureNotes1.pdf.