Modules, Splitting Sequences, and Direct Sums

Maria Ross

Department of Mathematics and Computer Science
University of Puget Sound

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Modules Over Rings

Definition

A *left R-module* M over a ring R is an abelian additive group together with a map $R \times M \to M$, denoted by $(r, m) \mapsto rm$ that satisfies the following properties for $r, s \in R$ and $m, n \in M$:

- $(r + s)m = rm + sm$
- $r(m + n) = rm + rn$
- $(rs)m = r(sm)$
- if $1 \in R$, then $1m = m$

Module Examples

- Vector Spaces are F-modules
- Rings are R-modules
- Abelian Groups are $\mathbb{Z}$-modules
- Ideals are R-modules

Matrices over a ring [1]

- R ring, $M_n(R)$ set of $n \times n$ matrices over R
- R acts on $M_n(R)$ by scalar multiplication, $r \mapsto rA$ for $r \in R$ and $A \in M_n(R)$
- $M_n(R)$ is an $R-$module

Basic Properties

Proposition

Consider a ring R and an R-module M. Then, for $r \in R$ and $m \in M$,

- $(r)0_M = 0_M$
- $(0_R)m = 0_M$
- $(-r)m = -(rm) = r(-m)$
- $(nr)m = n(rm) = r(nm)$ for all $n \in \mathbb{Z}$.

Submodules

Definition

A non-empty subset N of an $R-$module M is a *submodule* if for every $r, s \in R$ and $n, l \in N$, we have that $rn + sl \in N$.

- R ring of integers with ideal $6\mathbb{Z} \Rightarrow 6\mathbb{Z}$ is a $\mathbb{Z}$-module
- $12\mathbb{Z}$ subset of $6\mathbb{Z}$
- $x, y \in 12\mathbb{Z} \Rightarrow x = 12q$ and $y = 12r$ for some $q, r \in \mathbb{Z}$
- $ax + by = a(12q) + b(12r) = 12(aq) + 12(br) = 12(aq + br) \in 12\mathbb{Z}$
- So, $12\mathbb{Z}$ is a submodule of $6\mathbb{Z}$

Module Homomorphisms

Definition

If M and N are $R-$modules, a *module homomorphism* from M to N is a mapping $f : M \to N$ so that

(i) $f(m+n) = f(m) + f(n)$

(ii) $f(rm) = rf(m)$

for $m, n \in M$ and $r \in R$.

Quotient Structures

Proposition

Suppose R is a ring, M an $R-$module, and N a submodule of M. Then M/N, the quotient group of cosets of N, is an $R-$module.

Quotient Structures

Proof.

- M/N is an additive abelian group
- Define the action of R on M/N by $(r, m+N) \mapsto rm+N$
- By coset operations, for $r, s \in R$ and $m+N, l+N \in M/N$,

(i) $(r+s)(m+N) = r(m+N)+s(m+N) = (rm+N)+(sm+N)$

(ii) $r((m+N)+(l+N)) = r(m+l+N) = rm+rl+N = (rm+N)+(rl+N)$

(iii) $(rs)(m+N) = (rsm+N) = r(sm+N)$

(iv) if $1 \in R$, then $1(m+N) = 1m+N = m+N$.

Direct Sum of Modules

- I set of indices (finite or infinite)
- A *family* $(x_i, i \in I)$ is a function on I whose value at i is x_i

Definition [8]

The *external direct sum* of the modules M_i for $i \in I$ is $\bigoplus_{i \in I} M_i$, all families $(x_i, i \in I)$ with $x_i \in M_i$ such that $x_i = 0$ for all except finitely many i.

- Addition defined by $(x_i) + (y_i) = (x_i + y_i)$
- Scalar multiplication defined by $r(x_i) = (rx_i)$

Direct Sum of Modules

- For finite I, direct sum corresponds to direct product
- M, N are R-modules. Then

$$M \oplus N = \{(m, n) | m \in M, n \in N\}$$

- **Example:** Let $M = \mathbb{Z}_2$ and $N = \mathbb{Z}_3$ be $\mathbb{Z}$-modules, then

$$M \oplus N = \mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1)(1,2)\}.$$

Then $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$.

Internal Direct Sum

- R-module M with submodules M_1, M_2
- M is the *internal direct sum* of M_1 and M_2 if $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$
- Internal direct sum is isomorphic to external direct sum
- A *direct decomposition* of M is $M_1 \oplus M_2$ where $M \cong M_1 \oplus M_2$
- M is *indecomposable* if $M \ncong M_1 \oplus M_2$ for $M_1, M_2 \neq 0$

Free Modules and Cyclic Modules

- Modules with bases are called *free*
- M is a free module, then the *rank* of M is the number of elements in its basis
- An R-module M is *cyclic* if $\exists\ a \in M$ so $M = aR$

Proposition

A free R-module M is isomorphic to the direct sum of “copies” of R.

Free Modules

Example

Let M and N be free modules over $\mathbb{Z}$, with bases $B_M = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $B_N = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$. Then, $M = \{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{Z} \}$, and $N = \{ \begin{bmatrix} 2a \\ 2b \end{bmatrix} : a, b \in \mathbb{Z} \}$. If M and N were vector spaces, they would be isomorphic. However, because our scalar multiples are from a ring that does not have multiplicative inverses, N has no vectors with odd-parity entries. Thus, M and N are not isomorphic.

Torsion

Let R be an integral domain and M be an R-module:

- $x \in M$ is a *torsion element* if $rx = 0$ for $r \in R$, $r \neq 0$
- T, the set of all torsion elements of M, is a submodule of M
- T is called the *torsion submodule* of M
- if $T = M$, M is a *torsion module*
- if $T = \{0\}$, M is *torsion free*

Torsion Modules

Theorem [5]

Let T be a finitely generated torsion module over a PID R, and $\langle a_i \rangle$ ideals of R. Then T is isomorphic to the direct sum of cyclic torsion R-modules; that is,

$$T \cong R/\langle a_1 \rangle \oplus \cdots \oplus R/\langle a_m \rangle$$

for some m and nonzero $a_i \in R$.

Torsion Modules

Theorem

[5] If R is a PID, then every finitely generated R-module M is isomorphic to $F \oplus T$ where F is a finite free R-module and T is a finitely generated torsion R-module, which is of the form $T \cong \bigoplus_{j=i}^{m} R/\langle a_j \rangle$.

Exact Sequences

Suppose R is a ring, M_1, and M_2, M_3 are R-modules.

A sequence of module homomorphisms

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$

is *exact* if $\mathrm{im}(f_1) = \ker(f_2)$.

Short Exact Sequences

An exact sequence of the form

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$$

is called a *short exact sequence*.

- f_1 is injective
- f_2 is surjective

Examples of Short Exact Sequences

For any R-module M with submodule N, there is a short exact sequence

$$0 \longrightarrow N \xrightarrow{f_1} M \xrightarrow{f_2} M/N \longrightarrow 0$$

- $f_1 : N \to M$ defined by $f_1(n) = n$
- $f_2 : M \to M/N$ defined by $f_2(m) = m + N$

Examples of Short Exact Sequences

For ideals I and J of a ring R such that $I + J = R$, there is a short exact sequence

$$0 \longrightarrow I \cap J \xrightarrow{f_1} I \oplus J \xrightarrow{f_2} R \longrightarrow 0$$

- $f_1 : I \cap J \to I \oplus J$ is the map $f_1(x) = (x, -x)$
- $f_2 : I \oplus J \to R$ is addition where $\ker(f_2) = \{(x, -x) | x \in I \cap J\}$

Examples of Short Exact Sequences

L and M are $R-$modules with direct sum $L \oplus M$. There is a short exact sequence

$$0 \longrightarrow L \xrightarrow{f_1} L \oplus M \xrightarrow{f_2} M \longrightarrow 0$$

- $f_1 : L \to L \oplus M$ is the embedding of $l \in L$ into $L \oplus M$
- $f_2 : L \oplus M \to M$ is the projection of $x \in L \oplus M$ onto M, so that $f_2(f_1(l)) = 0$ for $l \in L$

Splitting Sequences

Definition [8]

A short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$$

is said to *split on the right* if there is a homomorphism $g_2 : M_3 \to M_2$ so that the function composition $f_2 \circ g_2 = 1$. The sequence *splits on the left* if there is a homomorphism $g_1 : M_2 \to M_1$ so that $f_1 \circ g_1 = 1$. A sequence that splits on the left and the right *splits*, and is called a *splitting sequence*.

The Five Lemma

Consider the following commutative diagram where both sequences of homomorphisms are exact:

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] & M_3 \arrow[r, "f_3"] \arrow[d, "h_3"'] & M_4 \arrow[r, "f_4"] \arrow[d, "h_4"'] & M_5 \arrow[d, "h_5"'] \\ L_1 \arrow[r, "g_1"'] & L_2 \arrow[r, "g_2"'] & L_3 \arrow[r, "g_3"'] & L_4 \arrow[r, "g_4"'] & L_5 \end{tikzcd}$$

- h_1 surjective
- h_2, h_4 bijective
- h_5 injective

$\Rightarrow$ h_3 is an isomorphism.

The Five Lemma Proof

Claim 1. If h_2 and h_4 are surjective and h_5 is injective, then h_3 is surjective.

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] & M_3 \arrow[r, "f_3"] \arrow[d, "h_3"'] & M_4 \arrow[r, "f_4"] \arrow[d, "h_4"'] & M_5 \arrow[d, "h_5"'] \\
L_1 \arrow[r, "g_1"'] & L_2 \arrow[r, "g_2"'] & L_3 \arrow[r, "g_3"'] & L_4 \arrow[r, "g_4"'] & L_5
\end{tikzcd}$$

- $x \in L_3 \Rightarrow g_3(x) \in L_4 \Rightarrow g_3(x) = h_4(y)$ for $y \in M_4$
- $g_4(g_3(x)) = g_4(h_4(y)) = 0$
- $g_4(h_4(y)) = h_5(f_4(y)) \Rightarrow g_4(g_3(x)) = h_5(f_4(y)) = 0$
- $h_5(f_4(y)) = 0 \Rightarrow f_4(y) \in \ker(h_5) \Rightarrow f_4(y) = 0.$
- $y \in \ker(f_4) = \mathrm{im}(f_3) \Rightarrow y = f_3(a)$ for some $a \in M_3$.

The Five Lemma Proof

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] & M_3 \arrow[r, "f_3"] \arrow[d, "h_3"'] & M_4 \arrow[r, "f_4"] \arrow[d, "h_4"'] & M_5 \arrow[d, "h_5"'] \\
L_1 \arrow[r, "g_1"'] & L_2 \arrow[r, "g_2"'] & L_3 \arrow[r, "g_3"'] & L_4 \arrow[r, "g_4"'] & L_5
\end{tikzcd}$$

- $g_3(x) = h_4(y) = h_4(f_3(a)) = g_3(h_3(a)) \Rightarrow x - h_3(a) \in \ker(g_3) = \mathrm{im}(g_2)$
- $x - h_3(a) = g_2(b)$ for $b \in L_2$
- $b = h_2(m)$ for $m \in M_2$, and
  $x - h_3(a) = g_2(b) = g_2(h_2(m)) = h_3(f_2(m))$
- $x - h_3(a) = h_3(f_2(m)) \Rightarrow x = h_3(a + f_2(m))$
- $x \in \mathrm{im}(h_3) \Rightarrow h_3$ is surjective.

The Five Lemma Proof

Claim 2. If h_2 and h_4 are injective and h_1 is surjective, then h_3 is injective.

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] & M_3 \arrow[r, "f_3"] \arrow[d, "h_3"'] & M_4 \arrow[r, "f_4"] \arrow[d, "h_4"'] & M_5 \arrow[d, "h_5"'] \\
L_1 \arrow[r, "g_1"'] & L_2 \arrow[r, "g_2"'] & L_3 \arrow[r, "g_3"'] & L_4 \arrow[r, "g_4"'] & L_5
\end{tikzcd}$$

- $a \in \ker(h_3) \Rightarrow h_4(f_3(a)) = g_3(h_3(a)) = g_3(0) = 0$
- $f_3(a) = 0 \Rightarrow a \in \ker(f_3) = \mathrm{im}(f_2) \Rightarrow a = f_2(z)$ for $z \in M_2$
- $0 = h_3(a) = h_3(f_2(z)) = g_2(h_2(z)) \Rightarrow h_2(z) \in \ker(g_2) = \mathrm{im}(g_1)$
- $h_2(z) = g_1(u)$, $u = h_1(v) \Rightarrow h_2(z) = g_1(h_1(v)) = h_2(f_1(v)) = h_2(z)$
- $z = f_1(v) \Rightarrow a = f_2(f_1(v)) = 0 \Rightarrow h_3$ is injective.

The Short Five Lemma

We consider a commutative diagram of short exact sequences, as below:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] & M_3 \arrow[r] \arrow[d, "h_3"'] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "g_1"'] & L_2 \arrow[r, "g_2"'] & L_3 \arrow[r] & 0
\end{tikzcd}$$

It follows directly from The Five Lemma that if h_1 and h_3 are isomorphisms, so is h_2.

Theorem [6]

Let R be a ring, and let M, N, and P be R-modules, with a short exact sequence of the form

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

Then, the following are equivalent:

(i) There is a homomorphism $f' : M \to N$ so that $f'(f(n)) = n$ for all $n \in N$; the sequence splits on the left.

(ii) There is a homomorphism $g' : P \to M$ so that $g'(g(p)) = p$ for all $p \in P$; the sequence splits on the right.

(iii) There is an isomorphism $\phi : M \to N \oplus P$, and the sequence splits.

We can illustrate this theorem by applying The Five Lemma to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "id"'] & M \arrow[r, "g"] \arrow[d, "\phi"'] & P \arrow[r] \arrow[d, "id"'] & 0 \\
0 \arrow[r] & N \arrow[r] & N \oplus P \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

Then, we can see that $M \cong N \oplus P$, and we have some insight into the structure of the original exact sequence. We can regard f as the embedding of N into M, and g as the projection of M onto P.

An Example of a Splitting Sequence

Consider the example from earlier with $\mathbb{Z}$-modules $\mathbb{Z}_2$ and $\mathbb{Z}_3$. There is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_6 \xrightarrow{g} \mathbb{Z}_3 \longrightarrow 0$$

- $f(x) = 3x$ for $x \in \mathbb{Z}_2$
- let $f' : \mathbb{Z}_6 \to \mathbb{Z}_2$ by $f'(y) = y \pmod 2$ for $y \in \mathbb{Z}_6$
- the sequence splits on the left

- $g(y) = 2y \pmod 3$ for $y \in \mathbb{Z}_6$
- let $g' : \mathbb{Z}_3 \to \mathbb{Z}_6$ by $g'(z) = 2z$ for $z \in \mathbb{Z}_3$
- the sequence splits on the right

An Example of a Splitting Sequence

We know the sequence splits, so there is a commutative diagram of the form

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "f"] \arrow[d] & \mathbb{Z}_6 \arrow[r, "g"] \arrow[d, "\phi"'] & \mathbb{Z}_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \arrow[r] & \mathbb{Z}_3 \arrow[r] & 0
\end{tikzcd}
$$

where ϕ is an isomorphism.

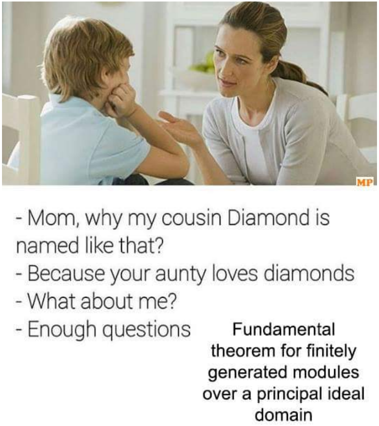
The Fundamental Theorem of Finitely Generated Modules Over Principal Ideal Domains

Suppose R is a PID and M is a finitely generated R-module. Then M is isomorphic to the direct sum of cyclic R-modules,

$$M \cong R/\langle d_1 \rangle \oplus R/\langle d_2 \rangle \oplus \cdots \oplus R/\langle d_n \rangle$$

where the $\langle d_i \rangle$ are ideals of R such that
$\langle d_n \rangle \subset \langle d_{n-1} \rangle \subset \cdots \subset \langle d_1 \rangle$ and $d_i | d_{i+1}$ for $1 \leq i \leq n$.

The End



Thank you.

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