Lie, Noether, and Lagrange

symmetries, and their relation to conserved quantities

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1 Introduction

The most beautiful result in all of physics is Noether's theorem; symmetries imply conservation laws. This result has informed our understanding of all of physics, from quantum field theory to general relativity and every domain in between [1]. The focus on symmetry in physics reflects on current research; to this day physicists are still working on Lie groups and their relation to the fundamental laws of nature. Symmetries provide the basis of our understanding of particle theory and led to the discovery of the Higg's boson [2]. Lie groups are a natural extension to standard group theory. Lie groups are infinite and there is a group element for each value of a continuous parameter [3]. Lie groups correspond to the symmetries of smooth surfaces [4]. The surface Noether's theorem cares about is the Lagrangian, which is the basis of Lagrangian mechanics.

Lagrangian mechanics is an alternative to the Newtonian formulation of the laws of physics. The Lagrangian is a function of the state of the system, which can then be used to determine the behavior of the system. The Lagrangian is itself a smooth surface, so we can use Lie groups to characterize the symmetries of the Lagrangian, which then neatly leads to Noether's theorem [1].

2 Lie Groups

Lie groups are infinite, continuous groups that represent the symmetries of differentiable manifolds [5]. While finite groups represent the discrete symmetries of objects such as regular polygons, Lie groups extend the language of group theory to the symmetries of objects such as spheres and hyperbolic paraboloids. The ability to formalize continuous symmetries will be invaluable when we try to explore the symmetries of the laws of physics [1].

2.1 Differentiable Manifolds

Before we can define Lie algebras, we must define the objects they act on. Manifolds are collection of points, and differentiable manifolds are the special case where the manifold looks flat around every point.

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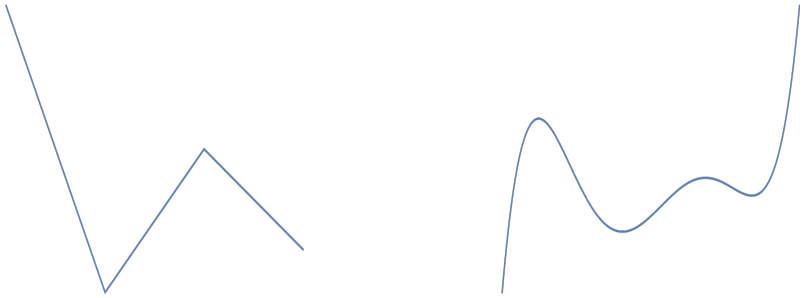

Figure 1: The manifold on the left is non-differentiable because there are points whose local area cannot be approximated as a plane. On the other hand, the manifold on the right is differentiable because every region can be approximated by a tangent line [6].

Definition 2.1 (Differentiable Manifold)**.** A differentiable manifold is a set of points in N-dimensional space connected in such a way that the set is smooth. Because they are smooth, differentiable manifolds approximate n-dimensional, for n less than N, Cartesian space around each point in the differentiable manifold [5].

In other words, differentiable manifolds are sets of points that are nice—there are no discontinuities or sharp points, and each point has neighbors arbitrarily close to it. Some examples of differentiable manifolds are

- the Cartesian spaces $\mathbb{R}^n$ and $\mathbb{C}^n$
- the n-dimensional sphere $\sum_{i=1}^{n} x_i^2 = r^2$ embedded in $\mathbb{R}^n$
- any smooth surface $\{(x_1, x_2, \ldots, x_n) \mid x_n = f(x_1, x_2, \ldots, x_{n-1})\}$ embedded in $\mathbb{R}^n$ or $\mathbb{C}^n$ [5].

For counterexamples, we can turn to the set of points defined by $y = |x|$ or fractals such as the Koch snowflake or Sierpinski gasket. The differences between differentiable and non-differentiable manifolds can be seen in Figure 1.

Because they are smooth, we can treat the area around a point in a differentiable manifold as we would the Cartesian space $\mathbb{R}^n$ [5]. For example, the surface of a sphere in $\mathbb{R}^3$ approximates $\mathbb{R}^2$. Thus, so long as we only care about the behavior of the differentiable manifold close to a point, the differentiable manifold is easy to describe—it is just our familiar Cartesian space!

2.2 Defining Lie Groups

Lie groups are defined to be groups of symmetries on differentiable manifolds where the symmetries are themselves differentiable with respect to some parameter ϵ [3, 7]. While groups such as dihedral or symmetric groups have finitely many elements, Lie groups have an infinite number of elements—one for each value of the continuous variable ϵ.

Definition 2.2 (Lie Group)**.** A Lie group is a set $G = \{\Gamma_\epsilon\}$ of smooth transformations $\Gamma_\epsilon : M \to M$ of a differentiable manifold M that depends on a continuous and sufficiently small parameter ϵ. Furthermore, Γ_0 is the identity operation and group multiplication is defined by $\Gamma_\alpha \Gamma_\beta = \Gamma_{\alpha+\beta}$ [3, 7].

Lie symmetries take points on a differentiable manifold to other points on the same differentiable manifold.

For example, the complex unit circle, which is the set of points $\{e^{it} \mid t \in \mathbb{R}\}$, has rotational symmetry, meaning that any rotation of the circle through an angle ϕ leaves the set that makes up the circle unchanged. Thus, we define a Lie group

$$G = \{\Gamma_\phi \mid \Gamma_\phi(e^{it}) = e^{it} \cdot e^{i\phi},\ \phi \in \mathbb{R}\}$$

that rotates the unit circle in the complex plane. Namely, $\Gamma_\phi(e^{it}) = e^{it} \cdot e^{i\phi} = e^{i(t+\phi)}$. We can check that G satisfies the conditions to be a Lie group. First note that $\Gamma_0(x) = e^0 \cdot x = x$ so Γ_0 is the identity element. Furthermore, $\Gamma_\phi \Gamma_\theta(x) = e^{i\phi} e^{i\theta} x = e^{(\phi+\theta)} x = \Gamma_{\phi+\theta}(x)$ for all ϕ and θ in $\mathbb{R}$.

We could reconstruct the same symmetry of the complex unit circle using only a linear expression in ϕ if we are willing to forgo the transformation being accurate for all ϕ. We achieve this by using Taylor expansions to note that

$$e^{i\phi} = \sum_{n=0}^{\infty} \frac{(i\phi)^n}{n!} = 1 + i\phi + \mathcal{O}(\phi^2).$$

Thus, for sufficiently small ϕ, we can neglect the term $\mathcal{O}(\phi^2)$.[2] Thus we can instead define the above Lie group on a circle as being the group $G = \{\Gamma_\phi \mid \Gamma_\phi(x) = (1 + i\phi) \cdot x,\ \phi \approx 0\}$. This leads us to the idea of infinitesimal generators.

2.3 Infinitesimal Generators

An infinitesimal generator is an element of the Lie group that is infinitesimally close to the identity operation. Using these generators, we can understand how Lie groups act on differentiable manifolds locally [7]. In general, we are able to use the same technique of taking the Taylor series of our transformation and ignoring all terms of order ϵ^2 or greater. Formally, we can write any element of the group $G = \{\Gamma_\epsilon\}$ as

$$\Gamma_\epsilon = \sum_{n=0}^{\infty} \frac{d^n \Gamma_\epsilon}{d\epsilon^n} \epsilon^n = \Gamma_0 + \frac{d\Gamma_\epsilon}{d\epsilon} \epsilon + \mathcal{O}(\epsilon^2) \approx \Gamma_0 + \frac{d\Gamma_\epsilon}{d\epsilon} \epsilon$$

for ϵ sufficiently close to 0 [4]. Note that we can reconstruct the entire operation Γ_ϵ given the term $\frac{d\Gamma_\epsilon}{d\epsilon}$. Thus, we know everything about the transformation Γ_ϵ as long as we know how it acts in vicinity of $\epsilon = 0$. For this reason, we call $\frac{d\Gamma_\epsilon}{d\epsilon}$ the infinitesimal generator of the Lie Group [1].

[2]The notation $\mathcal{O}(x^n)$, called "Big 'O' Notation", denotes any expression that is multiplied by x to the power n or greater. Thus, if x tends to 0 or is generally vanishingly small, we can say this term is effectively 0.

Definition 2.3 (Generator of a Lie Group)**.** The generator of a Lie group $G = \{\Gamma_\epsilon\}$ is the function $\frac{d\Gamma_\epsilon}{d\epsilon}$. The set $\{\frac{d\Gamma_\epsilon}{d\epsilon}\ \epsilon \mid \epsilon \approx 0\}$ is called the tangent algebra of the differentiable manifold.

The tangent algebra is, as the name would sugest, tangent to the differentiable manifold. While it is an algebra in its own right, if we want to use the tangent algebra to talk about the symmetries of the full manifold, we need to keep in mind that the tangent algebra only approximates the symetries of the full manifold for transformations close to the identity.

The generator presents two sorts of symmetries, one global and one local. We say that Γ_ϵ is a global symmetry of the differentiable manifold if Γ_ϵ leaves the differentiable manifold unchanged for all values of ϵ, and that it is a local symmetry if Γ_ϵ only preserves the differentiable manifold for small values of ϵ. For local symmetries, because ϵ must be vanishingly small, we can say that $\Gamma_\epsilon = \Gamma_0 + \frac{d\Gamma_\epsilon}{d\epsilon}\,\epsilon$. In some sense, local transformations move the differentiable manifold around in the region that it is locally like the Euclidean space that lies tangent to a point. This is the reason another name for the group formed by local symmetries is a tangent algebra [3].

3 Lagrangian Mechanics

Before we are able to use Lie symmetries to derive Noether's theorem, we must first take a detour into physics. Lagrangian mechanics are an alternate formulation of Newtonian physics. While Newton's laws describe how systems interact, they are not always the easiest tool to use.[3] Lagrangian mechanics give the same results as Newton, but can be significantly easier to use.

3.1 Phase Space

Phase space is the domain of the Lagrangian, and it covers all possible states of the system [5]. If you have a collection of N particles, then, in our 3-dimensional universe, you need $3N$ variables to describe the location of all the particles. Namely, the x_i, y_i, and z_i values of each particle indexed by $i \in \{1, 2, \ldots, N\}$. For the sake of compactness, instead of using x_i, y_i, and z_i to denote the location of a particle, we will use the generalized coordinate q_i for $i \in \{1, 2, \ldots, 3N\}$ [5]. This notation has the benefit that we are no longer confined to Cartesian coordinates, and are free to use any coordinate system we choose to use. The space described by these coordinates is called the configuration space.

However, given that Newton's second law is a second order differential equation, knowing the position of the system in configuration space is not enough to

[3] A typical example of this problem is the behavior of a satellite orbiting the Earth. Attempting to use Newton's laws results in needing to solve three linked and non-linear partial differential equations. However, if we instead use the Lagrangian formalism, the equations of motion quickly reduce to a single partial differential equation, which is much less frightening [8].

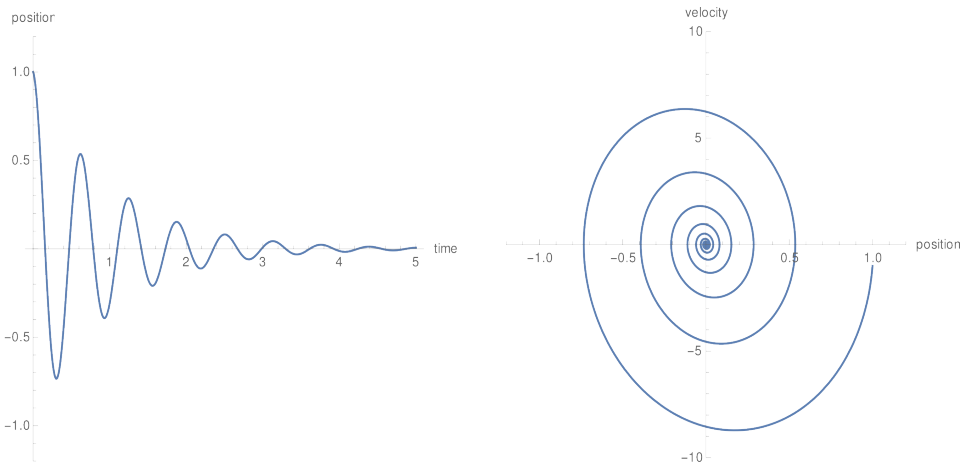

Figure 2: Depictions of the motion of a damped harmonic oscillator in configuration space (left) and phase space (right). Note the differences in the sort of axes; in configuration space, we deal with position versus time while in phase space, we deal with velocity versus position with the time evolution merely implied by the progression along the curve through phase space [6].

uniquely characterize all possible states of the system. If you want to know how long it would take for a ball to hit the ground, it's not enough to say that it is 5 meters above the ground, you also need to know its initial velocity. Similarly, to fully understand how any system will evolve in time, we also need information about the first time-derivatives of each of the coordinates. Thus, phase space includes the $3N$ pairs $(q_i, \dot{q}_i)$ [5]. As a result, if we have N particles and 3 dimensions, our phase space is $2 \cdot 3 \cdot N$ dimensional, with half the dimensions describing objects' locations and half specifying the objects' velocities.

Phase space is the set of all possible states of a system, and thus alone it tells us very little about the behavior of the system it describes. In order to know how an object behaves, we need its path through phase space as a function of time (see Figure 2). Using Newton's laws, this would consist of needing to solve a set of differential equation and plugging in the initial location in phase space to compute the behavior of the system. However, a more elegant solution is to define a function whose domain is the phase space, and whose shortest paths are the paths a physical system follows.

3.2 The Lagrangian

The Lagrangian is a function $\mathcal{L}(q_i, \dot{q}_i, t)$ for $i \in \{1, \ldots, 3N\}$ that, given a specific time t, assigns a real number to each point in phase space $(q_i, \dot{q}_i)$. As a result, the Lagrangian is a differentiable manifold in the $6N + 2$-dimensional space described by the phase space (dimension $6N$) combined with one dimension each for the time axis and the value of the Lagrangian. In classical mechanics, the Lagrangian takes

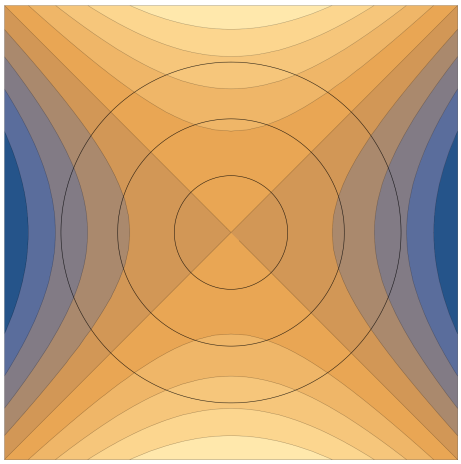

Figure 3: A contour plot of the Lagrangian of a simple harmonic oscillator. The manifold is scaled so that the scale on each axis is the same. Distance along the horizontal axis is position, while height represents velocities. The darker the region, the smaller the Lagrangian. The circles are allowed paths through phase space, and look like sine functions in configuration space [6].

the form of the difference between the kinetic energy and the potential energy. On the other hand, the potential energy function is any function $U(q_i, \dot{q}_i, t)$ [8]. Thus, symbolically,

$$\mathcal{L}(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i) - U(q_i, \dot{q}_i, t).$$

The exact definitions of the kinetic and potential energies depend on the specific physical system.

An example of a physical system that the Lagrangian is well suited for is that of an N-dimensional simple harmonic oscillator. We are given that

$$T = \sum_{i=1}^{N} \frac{1}{2} m \dot{q}_i^2 \quad \text{and} \quad U = \sum_{i=1}^{N} \frac{1}{2} k q_i^2.$$

Thus, using the definition of the Lagrangian, we find that, for a simple harmonic oscillator, the Lagrangian is

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2} (m \dot{q}_i^2 - k q_i^2),$$

which was nearly painless to calculate [8]. The Lagrangian for the simple harmonic oscillator, along with the physical paths can be found in Figure 3. While the Lagrangian is easy to calculate, we still need a way to translate this function into a path through phase space.

3.3 Shortest Paths

Nature wants to take the shortest route—as defined by the Lagrangian—through phase space that it can. In other words, nature is lazy. While in our 3-dimensional universe we are able to define length of a path $(x(t), y(t), z(t))$ as being

$$\text{Length}\,(x(t), y(t), z(t)) = \int_{t_1}^{t_2} (\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2)^{1/2}\, dt,$$

we need a different definition of length for phase space [8]. Here is where the Lagrangian becomes useful. The length of a path in phase space, otherwise known as the action S, is defined by adding up all the values of the Lagrangian along the path. Furthermore, because nature wants to take the shortest path, systems only take the path through phase space with the least action.

Definition 3.1 (Action)**.** The action of a Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$ along a path through phase space $(q_i(t), \dot{q}_i(t))$ is the quantity

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_i(t), \dot{q}_i(t), t)\, dt.$$

The principle of least action states that physically meaningful paths have the smallest possible action, or

$$\frac{dS}{d(q_i(t), \dot{q}_i(t))} = 0.$$

The derivative in the principle of least action is not one that we can evaluate using the normal techniques of differential calculus—there are too many variables that affect path length! Thus, in order to find the path that satisfies the principle of least action, we must borrow a result from the calculus of variations.

Theorem 3.1 (Euler-Lagrange Equations)**.** The action of the Lagrangian $\mathcal{L}$ is minimized along the path $(q_i(t), \dot{q}_i(t))$ through phase space if and only if

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L} q}{\partial \dot{q}_i} = 0.$$

Thus, in order to translate the Lagrangian into a solution, we must solve the above differential equation for each index i [8].

We now see how to find the behavior of a system using Lagrangian mechanics. First, we choose which coordinates q_i parametrize the system and how they relate to kinetic and potential energy. Next we calculate the Lagrangian, and solve the Euler-Lagrange equations to find the behavior of the system as time progresses. While this process can be simpler than using Newton's laws, it would be very nice if we could coerce the Lagrangian to tell us about the behavior of the system, even if we are unable to solve the Euler-Lagrange equations.

4 Noether's Theorem

Noether's theorem links conservation laws and Lie symmetries [9]. For every Lie symmetry of the Lagrangian there is a conserved quantity. This theorem is made possible with the realization that the Lagrangian is a differentiable manifold in phase-space. Since the Lagrangian is a differentiable manifold, we can easily ask what the symmetries are. Then, given the symmetries of the Lagrangian, we are able to find a conserved quantity.

4.1 Symmetries of the Lagrangian

Recall that the Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$ is a differentiable manifold and that Lie symmetries are operations on differentiable manifolds that preserve the manifold's structure. If we have a set of operations Γ_ϵ that are continuous transformations of the Lagrangian such that $\Gamma_\epsilon(\mathcal{L}) = \mathcal{L}$, then we have a Lie group on the Lagrangian. We will find that the existence of such groups implies there are conserved quantities of the system.

If $\Gamma_\epsilon(\mathcal{L}) = \mathcal{L}$, then we can think of the operation Γ_ϵ as acting on the domain of $\mathcal{L}$ [7]. Because there is a function equality, the transformation cannot redefine the function $\mathcal{L}$. Thus, we can write

$$\Gamma_\epsilon(\mathcal{L}(q_i, \dot{q}_i, t)) = \mathcal{L}(Q_i(q_j, \dot{q}_j, t, \epsilon), \dot{Q}_i(q_j, \dot{q}_j, t, \epsilon), T(q_j, \dot{q}_j, t, \epsilon)).$$

We have written the Lie symmetry of the Lagrangian in the form of continuous transformations of each of the variables. Each point in phase space is sent to another point in phase space.

While the above is true for both global and local symmetries, it would be nice to have the transformation in a form that is easily applicable to local transformations. We proceed by the same technique as we used when we first encountered local symmetries—we will expand each transformation using a Taylor series in ϵ. Thus, for local symmetries,

$$\Gamma_\epsilon(\mathcal{L}(q_i, \dot{q}_i, t)) = \mathcal{L}(q_i + \zeta_i\epsilon, \dot{q}_i + \dot{\zeta}_i\epsilon, t + \tau\epsilon).$$

Where ζ_i, $\dot{\zeta}_i$, and τ are defined by

$$\zeta_i = \frac{\partial Q_i}{\partial \epsilon}\Big|_{\epsilon=0} \qquad \dot{\zeta}_i = \frac{\partial \dot{Q}_i}{\partial \epsilon}\Big|_{\epsilon=0} \qquad \tau = \frac{\partial T}{\partial \epsilon}\Big|_{\epsilon=0}.$$

Local transformations are much nicer than global transformations—we no longer need to worry about higher powers of ϵ.

4.2 Rund-Trautman Identity

In order to prove Noether's theorem, we need to prove an important lemma—the Rund-Trautman identity.

Theorem 4.1 (The Rund-Trautman Identity)**.** Let Γ_ϵ be a local transformation of the Lagrangian $\Gamma_\epsilon(\mathcal{L}) = \mathcal{L}'$ and let S and S' be the actions of the Lagrangian $\mathcal{L}$ and $\mathcal{L}'$ where

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)\, dt \quad \text{and} \quad S' = \int_{t_1}^{t_2} \mathcal{L}'(q_i, \dot{q}_i, t)\, dt.$$

Then, if the Lagrangian $\mathcal{L}$ is locally invariant under Γ_ϵ to the first order of ϵ, or, in other words, if

$$\mathcal{L}(q_i, \dot{q}_i, t) = \Gamma_\epsilon(\mathcal{L}(q_i, \dot{q}_i, t)) + \mathcal{O}(\epsilon^2),$$

then the identity

$$\frac{\partial \mathcal{L}}{\partial q_i}\zeta_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{\zeta}_i + \frac{\partial \mathcal{L}}{\partial t}\tau - \mathcal{H}\dot{\tau} = 0$$

holds, where $\mathcal{H} = \dot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L}$ is called the Hamiltonian. [1, 10]

Proof. If $S(q_i + \zeta_i\,\epsilon, t + \tau\,\epsilon) - S(q_i, t)$, then $S(q_i + \zeta_i\,\epsilon, t + \tau\,\epsilon) - S(q_i, t) = S' - S = 0$. If we go back to the definition of the action as the integral of the Lagrangian and use the chain rule to collect terms inside a single integral, we write

$$\int_{t_1}^{t_2} \mathcal{L}' \frac{dt'}{dt} - \mathcal{L}dt = \mathcal{O}(\epsilon^2)$$

While this situation would normally be intractable because the integral is definite, we never actually defined t_1 or t_2; they're arbitrary! Because the bounds of integration are arbitrary, we can take the integral over any range we desire, and still require it to evaluate to $\mathcal{O}(\epsilon^2)$. The only integrand for which this holds is the 0 function. Thus,

$$\mathcal{L}' \frac{dt'}{dt} - \mathcal{L}dt = 0.$$

If we take the partial derivative of this expression with respect to ϵ and evaluate the result at $\epsilon = 0$, we find that

$$\frac{\partial \mathcal{L}}{\partial q_i}\zeta_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{\zeta}_i + \frac{\partial \mathcal{L}}{\partial t}\tau - \Big(\dot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L}\Big)\dot{\tau} = 0,$$

which results in the Rund-Trautman identity if we substitute $\mathcal{H}$ in for $\dot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L}$ [1, 10]. □

The Rund-Trautman identity provides a differential equation corresponding to local Lie symmetries of the Lagrangian. Now that we have this relation, it is a simple step to go from this equation to Noether's Theorem.

4.3 Noether's Theorem

We are now able to combine the Rund-Trautman identity and the Euler-Lagrange equations to derive Noether's theorem. Recall that the Rund-Trautman identity tells us about which symmetries the Lagrangian, and thus the action, are invariant under, and that the Euler-Lagrange equations describe extremal actions [10]. If we can combine these two theorems, Noether's theorem falls out.

Theorem 4.2 (Noether's Theorem)**.** Let Γ_ϵ is a local symmetry of the Lagrangian. Then, if $\mathcal{L}$ is the Lagrangian, $\mathcal{H}$ is the Hamiltonian, we can conclude that

$$\frac{\partial}{\partial t}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\zeta_i - \mathcal{H}\tau\Big) = 0,$$

or, in other words, the quantity $\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\zeta_i - \mathcal{H}\tau$ is conserved.

Proof. Because Γ_ϵ is a symmetry of the Lagrangian, the Rund-Trautman identity holds. Thus,

$$\frac{\partial \mathcal{L}}{\partial x_i}\zeta_i + \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\dot{\zeta}_i + \frac{\partial \mathcal{L}}{\partial t}\tau - \mathcal{H}\dot{\tau} = 0.$$

Because we only care about the path over the Lagrangian that is minimal, we can use the Euler-Lagrange equations and the definition of the Hamiltonian to simplify the above expression. Thus, because

$$\frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}_i},$$

we can rewrite the Rund-Trautman identity as

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\zeta_i + \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\dot{\zeta}_i - \frac{\partial \mathcal{H}}{\partial t}\tau - \mathcal{H}\dot{\tau} \\ &= \frac{\partial}{\partial t}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\zeta_i\Big) - \frac{\partial}{\partial t}\Big(\mathcal{H}\tau\Big) \\ &= \frac{\partial}{\partial t}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\zeta_i - \mathcal{H}\tau\Big). \end{aligned}$$

Thus the quantity $\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\zeta_i - \mathcal{H}\tau$ is conserved and we conclude that Noether's theorem holds; symmetries imply conservation laws [1, 10]. □

It is difficult to overstate how impressive this theorem is. The fact that symmetries of the laws of nature imply conservation laws is, in some sense, absurd. There is no hint from the definition of the Lagrangian that this theorem arises from the laws of physics.

5 Applications

One of the prototypical examples of conservation laws are the laws of conservation of energy and of momentum. These conservation laws have practical applications in all fields of physics. Conservation of energy is the foundation for all of thermodynamics and without conservation of momentum, it would be practically impossible to understand collisions. Using Lie symmetries, we can derive these two conservation laws [11].

5.1 Conservation of Energy

Suppose the Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$ is only a function of q_i and $\dot{q}_i$. In other words, the Lagrangian is time-invariant. Then the Lagrangian is trivially a symmetry of the Lie transformation

$$\Gamma_\epsilon(\mathcal{L}(q_i, \dot{q}_i, t)) = \mathcal{L}(q_i, \dot{q}_i, t - \epsilon).^4$$

Thus, we find that

$$\zeta_i = \frac{\partial Q_i}{\partial \epsilon}\Big|_{\epsilon=0} = 0 \qquad \dot{\zeta}_i = \frac{\partial \dot{Q}_i}{\partial \epsilon}\Big|_{\epsilon=0} = 0 \qquad \tau = \frac{\partial T}{\partial \epsilon}\Big|_{\epsilon=0} = -1.$$

Given these values of ζ, $\dot{\zeta}$, and τ, we are able to evaluate the expression $\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\zeta_i - \mathcal{H}\tau - F$, which we know is conserved. Thus, if the Lagrangian does not depend on time, the quantity $\mathcal{H} = \dot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L}$ is conserved. Symbolically,

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\partial}{\partial t}(\dot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L}) = 0.$$

In classical mechanics, the Lagrangian is of the form

$$\mathcal{L} = T - U = \frac{1}{2} m_i \dot{q}_i{}^2 - U(q_i),$$

so the Hamiltonian is

$$\begin{aligned}
\mathcal{H} &= \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L} \\
&= m_i \dot{q}_i^2 - \Big(\frac{1}{2} m_i \dot{q}_i{}^2 - U(q_i)\Big) \\
&= \frac{1}{2} m_i \dot{q}_i{}^2 + U(q_i) \\
&= T + U.
\end{aligned}$$

Thus, as long as the Lagrangian is time-independent, the Hamiltonian, better known as the energy, is conserved. This is a spectacular result; the law of conservation of energy stems from systems being time independent [11].

[4] The factor in front of the ϵ doesn't actually matter, but the results are both more elegant and striking if we subtract ϵ instead of adding it.

5.2 Conservation of Momentum

While the previous result stems from time independence, for our next result, we shall assume that the Lagrangian does not depend on the position q_i. Thus, the Lie symmetry of the Lagrangian is of the form

$$\Gamma_{\epsilon_i}(\mathcal{L}(q_i, \dot{q}_i, t)) = \mathcal{L}(q_i + \epsilon_i, \dot{q}_i, t).$$

Following a similar path as the above derivation, we find the quantities

$$\zeta_i = \frac{\partial Q_i}{\partial \epsilon}\Big|_{\epsilon=0} = 1 \qquad \dot{\zeta}_i = \frac{\partial \dot{Q}_i}{\partial \epsilon}\Big|_{\epsilon=0} = 0 \qquad \tau = \frac{\partial T}{\partial \epsilon}\Big|_{\epsilon=0} = 0.$$

Once again, we know that the quantity $\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\zeta_i - \mathcal{H}\tau$ is conserved, so given $\tau = 0$ and $\zeta = 1$,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = p_i$$

where we define p_i to be the generalized momentum. Note that, while conservation of energy concerns a scalar quantity, the conservation of momentum is a vector quantity because of the subscript letting the momentum point in multiple directions [11]. These results are impressive—as long as our system is time and space independent, we can use the laws of conservation of momentum and energy.

6 Conclusion

Lie groups are a very powerful tool. Using Noether's theorem, we find that Lie symmetries of the Lagrangian imply conserved quantities. Lie symmetries are powerful because they can describe the symmetries of physical laws. The beauty of Noether's theorem is that groups—objects that obey simple rules and are very much the result of pure mathematics—imply the experimentally verified conservation laws. This connection between pure mathematics and the laws that govern reality is beautiful.

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