Math 290 Name: Key Dr. Beezer
Exam 1 Fall 2019
Chapter SLE

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices, except in the question that asks you to row-reduce without Sage. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Solve the following system of linear equations and express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} 3x_1 + x_2 + 7x_3 + 3x_4 &= 13 \\ -x_1 \qquad - 4x_3 + x_4 &= 3 \\ -2x_1 - x_2 - 2x_3 - 5x_4 &= -20 \\ x_1 + x_2 - 3x_3 + 8x_4 &= 31 \end{aligned}$$

Augmented Matrix

$$\left[\begin{array}{cccc|c} 3 & 1 & 7 & 3 & 13 \\ -1 & 0 & -4 & 1 & 3 \\ -2 & -1 & -2 & -5 & -20 \\ 1 & 1 & -3 & 8 & 31 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

no pivot column in column 5, so RCLS ⇒ consistent
Then FVCS implies $n-r = 4-4 = 0$ free variables, so one solution.

$$\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix}\right\}$$

2. Without using Sage, find a matrix B in reduced row-echelon form which is row-equivalent to A. It is especially important to show all of your work, so it is clear you have not used Sage. (15 points)

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & 2 \\ 2 & 3 & 7 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -3 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{-1R_2} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow[-R_2+R_3]{-2R_2+R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Is the matrix B singular or not? Provide a justification for your answer. (15 points)

$$B = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 3 & -2 & 8 & 1 \\ -1 & 0 & 1 & -6 \\ 2 & -1 & 8 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4 \text{ the } 4\times 4 \text{ identity matrix}$$

By Theorem NMRRI, B is nonsingular.

4. Determine the solutions to the homogeneous system $\mathcal{LS}(B, \mathbf{0})$ and express them compactly as a (infinite) set of column vectors. Then do the necessary algebra to convince someone that your solution set is correct. (15 points)

$$B = \begin{bmatrix} -1 & -2 & 1 & -2 \\ -2 & -5 & 3 & -8 \\ 1 & 2 & 0 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

If we augment these matrices with the zero vector we have equations

$x_1 = 3x_4$

$x_2 = -x_4$ where x_4 is a free variable. $S = \left\{ \begin{bmatrix} 3x_4 \\ -x_4 \\ 3x_4 \\ x_4 \end{bmatrix} \middle| \; x_4 \in \mathbb{C} \right\}$

$x_3 = 3x_4$

Now, evaluate the original equations with this generic solution

Equation 1: $-x_1 - 2x_2 + x_3 - 2x_4$

$= -(3x_4) - 2(-x_4) + (3x_4) - 2x_4 = 0$ ✓

And similarly for Equation 2 and Equation 3.

5. Suppose that A is a nonsingular 3×3 matrix, and C is a singular 3×3 matrix. Let $\mathbf{0}$ and $\mathbf{c}$ be the vectors below. For each of the four linear systems, say as much as you can about the solution set. (20 points)

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$$

(a) $\mathcal{LS}(A, \mathbf{0})$

Nonsingular coefficient matrix implies unique solution. That unique solution is the trivial solution since the system is homogeneous

(b) $\mathcal{LS}(A, \mathbf{b})$

Non singular coefficient matrix implies a unique solution. (We know it is <u>not</u> the trivial solution.)

(c) $\mathcal{LS}(C, \mathbf{0})$

$\left.\begin{array}{l}\text{Homogeneous system} \Rightarrow \text{consistent.} \\ \text{Singular coefficient matrix} \Rightarrow \underline{\text{not}} \text{ unique solution}\end{array}\right\} \Rightarrow$ Infinitely many solutions, trivial solution is one

(d) $\mathcal{LS}(C, \mathbf{b})$

Singular coefficient matrix $\Rightarrow$ not unique solution.

So no solutions, or infinitely many. We can't tell which.

6. Suppose we have a homogeneous linear system that has coefficient matrix A, $\mathcal{LS}(A, \mathbf{0})$. Suppose further that every column of A is the same (i.e. identical). Prove that the system has infinitely many solutions. (15 points)

Two outlines:

1) $[A \mid \underset{\sim}{0}]$ has every row a multiple of every other. When "row-reduced", $r=1$, so there are zero rows and free variables.

2) $x_1 = 1, x_2 = -1, x_3 = 0, \ldots, x_n = 0$ is a solution. $\underset{\sim}{x} = \underset{\sim}{0}$ is a solution. Two solutions implies we must have infinitely many.

Q: Do you see that this problem has a slight problem?