Math 290 Name: Key Dr. Beezer
Exam 2 Fall 2019
Chapter V

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Is the set of vectors $T \subset \mathbb{C}^5$ linearly independent? Justify your answer. (15 points)

$$T = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 4 \\ 4 \\ -8 \end{bmatrix} \right\}$$

Theorem LIVRN suggests a matrix w/ column vectors as columns of the matrix

$$A = \begin{bmatrix} 0 & -1 & -1 & -3 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 4 \\ 1 & 0 & -1 & 4 \\ -1 & -1 & -1 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the RREF we have
$n = 4$ columns
$r = 4$ pivots
& so T is linearly independent.

2. Is the vector $\mathbf{w}$ an element of the span of R, $\langle R \rangle$? Justify your answer. (15 points)

$$\mathbf{w} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \qquad T = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -8 \end{bmatrix} \right\}$$

Scalars $\alpha_1, \alpha_2, \alpha_3$ so that $\alpha_1 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} -2 \\ 4 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$? Defn SSCV

Solution to system w/ augmented matrix? Theorem SLSLC

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ -3 & -2 & 4 & -4 \\ 0 & 4 & -8 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

By Theorem RCLS this system is inconsistent.

no solution $\Rightarrow$ $\underset{\sim}{w} \notin \langle T \rangle$.

3. For the following system of linear equations express a typical (generic) solution using the "vector form of the solutions" as described in Theorem VFSLS and related examples. (10 points)

$$\begin{aligned} 3x_1 + x_2 + 4x_3 - x_4 + 6x_5 &= 5 \\ 3x_1 - 2x_2 - 3x_3 + 6x_4 - 3x_5 &= 4 \\ -x_1 - x_2 - 3x_3 + 2x_4 - 4x_5 &= -2 \end{aligned}$$

row-reduce augmented matrix of the system

$$\left[\begin{array}{ccccc|c} 3 & 1 & 4 & -1 & 6 & 5 \\ 3 & -2 & -3 & 6 & -3 & 4 \\ -1 & -1 & -3 & 2 & -4 & -2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{\wedge}{x} = \begin{bmatrix} \\ \\ \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \\ \\ \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

4. Find a linearly independent set R so that $\langle R \rangle = \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the null space of A. Provide justifications of the requested properties. (15 points)

$$A = \begin{bmatrix} -1 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 3 & -1 \\ -2 & 1 & 3 & 4 & 7 \end{bmatrix}$$

Theorem BNS Suggest row-reducing & solving the homogeneous system $LS(A, \underset{\sim}{0})$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix}$$

x_4 & x_5 are free

$$N(A) = \left\langle \left\{ \begin{bmatrix} \\ \\ \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

5. Suppose $W = \langle S \rangle$. Find a linearly independent set T so that $W = \langle T \rangle$, with justifications of the requested properties. (15 points)

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix} \right\}$$

Theorem BS suggests making the vectors the columns of a matrix, row-reducing and identifying pivot columns.

$$A = \begin{bmatrix} 1 & 3 & 3 & 7 & 4 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 2 & 4 & 7 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -3 & 0 & -1 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$D = \{1, 2, 4\}$ indices of pivot columns

"Keep" column vectors 1, 2 & 4;

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 7 \end{bmatrix} \right\}$$

6. Suppose we have a scalar $\alpha \in \mathbb{C}$ and a vector $\mathbf{x} \in \mathbb{C}^n$. Prove that $\overline{\alpha \mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$. (15 points)

For $1 \le i \le n$,

$$
\begin{aligned}
[\overline{\alpha \underset{\sim}{x}}]_i &= \overline{[\alpha \underset{\sim}{x}]_i} && \text{Defn CCCV} \\
&= \overline{\alpha [\underset{\sim}{x}]_i} && \text{Defn CVSM} \\
&= \overline{\alpha}\ \overline{[\underset{\sim}{x}]_i} && \text{Theorem CCRM} \\
&= \overline{\alpha} [\overline{\underset{\sim}{x}}]_i && \text{Defn CCCV} \\
&= [\overline{\alpha}\,\overline{\underset{\sim}{x}}]_i && \text{Defn CVSM}
\end{aligned}
$$

So by Defn CVE, $\overline{\alpha \underset{\sim}{x}} = \overline{\alpha}\ \overline{\underset{\sim}{x}}$

7. Suppose we have vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{C}^n$ such that $\mathbf{u}$ is orthogonal to $\mathbf{v}$, and $\mathbf{u}$ is orthogonal to $\mathbf{w}$. Prove that $\mathbf{u}$ is orthogonal to $8\mathbf{v} + 3\mathbf{w}$. (15 points)

Check $\langle \underset{\sim}{u}, 8\underset{\sim}{v}+3\underset{\sim}{w} \rangle$

$$
\begin{aligned}
&= \langle \underset{\sim}{u}, 8\underset{\sim}{v} \rangle + \langle \underset{\sim}{u}, 3\underset{\sim}{w} \rangle && \text{theorem IPVA} \\
&= 8 \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle + 3 \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle && \text{theorem IPSM} \\
&= 8 \cdot 0 + 3 \cdot 0 && \text{Hypothesis} \\
&= 0
\end{aligned}
$$

So by Definition OV, $\underset{\sim}{u}$ & $8\underset{\sim}{v}+3\underset{\sim}{w}$ are orthogonal.