Math 290 Name: Key Dr. Beezer
Exam 4 Fall 2019
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Determine if the subset $T$ of the vector space of polynomials with degree at most 3, $P_3$, is linearly independent. (15 points)

$T = \{x^3 - 5x^2 + 4x - 2, x^3 - 5x^2 + 3x - 2, -6x^3 + 4x^2 + x + 7\}$ Start w/ a RLD.

$$\alpha_1(x^3-5x^2+4x-2) + \alpha_2(x^3-5x^2+3x-2) + \alpha_3(-6x^3+4x^2+x+7) = \mathbf{0}$$

$$(\alpha_1+\alpha_2-6\alpha_3)x^3 + (-5\alpha_1-5\alpha_2+4\alpha_3)x^2 + (4\alpha_1+3\alpha_2+\alpha_3)x + (-2\alpha_1-2\alpha_2+7\alpha_3)$$
$$= 0x^3+0x^2+0x+0$$

Equate coefficients to obtain homogeneous system with coefficient matrix

$$\begin{bmatrix} 1 & 1 & -6 \\ -5 & -5 & 4 \\ 4 & 3 & 1 \\ -2 & -2 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Only solution: $\alpha_1 = \alpha_2 = \alpha_3 = 0$

So, yes, $T$ is linearly independent.

2. Does the set $R$ span the vector space of $2 \times 2$ matrices, $M_{22}$? That is, does $\langle R \rangle = M_{22}$? (15 points)

$R = \left\{ \begin{bmatrix} 4 & 2 \\ 3 & -5 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ 4 & -4 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & -8 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 6 & 3 \end{bmatrix} \right\}$

Quick solution: size of $R = 4$, $\dim(M_{22}) = 4$, show $R$ linearly independent, then Theorem G implies $R$ does span $M_{22}$.

Direct Solution: $\alpha_1, \alpha_2, \alpha_3, \alpha_4$? so that

$$\alpha_1 \begin{bmatrix} 4 & 2 \\ 3 & -5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 5 & 3 \\ 4 & -4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 & 0 \\ 2 & -8 \end{bmatrix} + \alpha_4 \begin{bmatrix} 7 & 8 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

arbitrary element of $M_{22}$

$$\begin{bmatrix} 4\alpha_1+5\alpha_2+3\alpha_3+7\alpha_4 & 2\alpha_1+3\alpha_2+0\alpha_3+8\alpha_4 \\ 3\alpha_1+4\alpha_2+2\alpha_3+6\alpha_4 & -5\alpha_1-4\alpha_2-8\alpha_3+3\alpha_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrix equality gives system with augmented matrix

$$\left[\begin{array}{cccc|c} 4 & 5 & 3 & 7 & a \\ 2 & 3 & 0 & 8 & b \\ 3 & 4 & 2 & 6 & c \\ -5 & -4 & -8 & 3 & d \end{array}\right]$$

coefficient matrix row-reduces to $I_4$, so nonsingular. By Theorem NMUS

$\rightarrow$ there is always a solution.

So, yes, $R$ does span $M_{22}$.

3. The set $W$ is a subspace of the vector space of polynomials with degree at most 2, $P_2$. (You may assume this.) (40 points)

$W = \{a + bx + cx^2 \,|\, a + 2b - c = 0\}$

(a) Prove that the dimension of $W$ is 2, that is, $\dim(W) = 2$.

Need a basis, so get a spanning set first

$a + 2b - c = 0 \Rightarrow a = -2b + c$

$W = \{(-2b+c) + bx + cx^2 \mid b, c \in \mathbb{C}\} = \{b(-2+x) + c(1+x^2) \mid b, c \in \mathbb{C}\}$

$= \langle\{-2+x, 1+x^2\}\rangle$ $\quad S = \{-2+x, 1+x^2\}$ spans $W$.

A proof that $S$ is linearly independent is straightforward (see #1).

$\dim W =$ size of $S = 2$

(b) Does $K = \{2 + x + 4x^2\}$ span $W$?

Size of $K = 1 < 2 = \dim W$

Goldilocks says $K$ does not span $W$.

(c) Is $L = \{5 + 5x^2, x + 2x^2, 4 + x + 6x^2\}$ a linearly independent subset of $W$?

Size of $L = 3 > 2 = \dim W$

Goldilocks says $L$ is linearly dependent.

(d) Is $B = \{-1 + 3x + 5x^2, 3 - 2x - x^2\}$ a basis for $W$? B has the "right" size, but this is not enough. Check that B is linearly independent (see #1). Then Goldilocks says B spans $W$. Thus B is a basis of $W$.

4. Prove that $U$ is a subspace of the vector space $\mathbb{C}^3$, by using the three-part test of Theorem TSS. (15 points)

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, 3a + b + c = 0 \right\}$$

① $U \neq \emptyset$? Easiest / best to check if $\mathbf{0} \in U$.

$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $3 \cdot 0 + 0 + 0 = 0$ and $\mathbf{0} \in U$.

② Additive Closure

Suppose $\mathbf{x} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \in U$ and $\mathbf{y} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \in U$. Know $\begin{matrix} 3a_1 + b_1 + c_1 = 0 \\ 3a_2 + b_2 + c_2 = 0 \end{matrix}$.

$\mathbf{x} + \mathbf{y} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix}$ Examine $3(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2)$

$$\begin{aligned} &= 3a_1 + 3a_2 + b_1 + b_2 + c_1 + c_2 \\ &= (3a_1 + b_1 + c_1) + (3a_2 + b_2 + c_2) \\ &= 0 + 0 = 0 \end{aligned}$$

So $\mathbf{x} + \mathbf{y} \in U$.

③ Scalar Closure

Suppose $\alpha \in \mathbb{C}$, $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in U$. Know $3a + b + c = 0$.

$\alpha \mathbf{x} = \begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix}$. Examine $3(\alpha a) + \alpha b + \alpha c = \alpha(3a) + \alpha b + \alpha c = \alpha(3a + b + c) = \alpha \cdot 0 = 0$

So $\alpha \mathbf{x} \in U$

5. Suppose that $\alpha, \beta \in \mathbb{C}$, $V$ is a vector space, $\mathbf{v} \in V$, $\mathbf{v} \neq \mathbf{0}$, and $\alpha \mathbf{v} = \beta \mathbf{v}$. Prove that $\alpha = \beta$. (15 points)

This is Exercise VS.T23, and there is a solution there. Here is a different solution, by contradiction. Assume $\alpha \neq \beta$. Then $\alpha - \beta \neq 0$.

Then

$$\begin{aligned} \mathbf{v} &= 1\mathbf{v} \\ &= \left(\frac{1}{\alpha - \beta}\,\alpha - \beta\right)\mathbf{v} && \text{note: } \alpha - \beta \neq 0 \text{ so division is OK} \\ &= \frac{1}{\alpha - \beta}\left((\alpha - \beta)\mathbf{v}\right) \\ &= \frac{1}{\alpha - \beta}\left(\alpha\mathbf{v} - \beta\mathbf{v}\right) \\ &= \frac{1}{\alpha - \beta}\left(\alpha\mathbf{v} - \alpha\mathbf{v}\right) && \text{Hypothesis} \\ &= \frac{1}{\alpha - \beta}\,\mathbf{0} \\ &= \mathbf{0} && \text{But this contradicts } \mathbf{v} \neq \mathbf{0}. \end{aligned}$$