Math 290 Name: Key Dr. Beezer
Exam 6 Fall 2019
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Prove that S is a linear transformation. (15 points)

$$S\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad S\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}b+c\\a+2c\end{bmatrix}$$

1) $$S(\underset{\sim}{x}+\underset{\sim}{y}) = S\left(\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}+\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}\right) = S\left(\begin{bmatrix}a_1+a_2\\b_1+b_2\\c_1+c_2\end{bmatrix}\right)$$

$$= \begin{bmatrix}b_1+b_2+c_1+c_2\\a_1+a_2+2(c_1+c_2)\end{bmatrix} = \begin{bmatrix}b_1+c_1+b_2+c_2\\a_1+2c_1+a_2+2c_2\end{bmatrix} = \begin{bmatrix}b_1+c_1\\a_1+2c_1\end{bmatrix}+\begin{bmatrix}b_2+c_2\\a_2+2c_2\end{bmatrix}$$

$$= S(\underset{\sim}{x})+S(\underset{\sim}{y})$$

2) $$S(\alpha\underset{\sim}{x}) = S\left(\alpha\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = S\left(\begin{bmatrix}\alpha a\\\alpha b\\\alpha c\end{bmatrix}\right) = \begin{bmatrix}\alpha b+\alpha c\\\alpha a+2(\alpha c)\end{bmatrix} = \begin{bmatrix}\alpha(b+c)\\\alpha(a+2c)\end{bmatrix}$$

$$= \alpha\begin{bmatrix}b+c\\a+2c\end{bmatrix} = \alpha S(\underset{\sim}{x})$$

2. For the linear transformation S in the previous problem, compute the preimage $S^{-1}\left(\begin{bmatrix}1\\4\end{bmatrix}\right)$. (15 points)

Solve $S\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}1\\4\end{bmatrix}$ $\quad \begin{aligned} b+c&=1\\ a+2c&=4\end{aligned}$ $\quad \left[\begin{array}{ccc|c}0&1&1&1\\1&0&2&4\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c}1&0&2&4\\0&1&1&1\end{array}\right]$

swap rows!

$$S^{-1}\left(\begin{bmatrix}1\\4\end{bmatrix}\right) = \left\{\begin{bmatrix}a\\b\\c\end{bmatrix}\,\middle|\, a=4-2c,\ b=1-1c\right\}$$

$$= \left\{\begin{bmatrix}4-2c\\1-c\\c\end{bmatrix}\,\middle|\, c\in\mathbb{C}\right\} = \left\{\begin{bmatrix}4\\1\\0\end{bmatrix}+c\begin{bmatrix}-2\\-1\\1\end{bmatrix}\,\middle|\, c\in\mathbb{C}\right\}$$

$$= \underbrace{\begin{bmatrix}4\\1\\0\end{bmatrix}}_{\text{"particular solution"}} + \underbrace{\left\langle\left\{\begin{bmatrix}-2\\-1\\1\end{bmatrix}\right\}\right\rangle}_{K(S)}$$

Theorem KPI in action

3. Consider the linear transformation R from the vector space of 2×2 matrices, M_{22}, to the vector space of polynomials with largest degree 3, P_3. (20 points)

$$R\colon M_{22} \to P_3, \quad R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (-2a - b - 5c + d) + (-a - b - 4c)x + (2b + 7c + 3d)x^2 + (a + b + 8c + 4d)x^3$$

(a) Compute the kernel of R, $\mathcal{K}(R)$.

$R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 + 0x + 0x^2 + 0x^3$

$\Rightarrow$ homogeneous system w/ coefficient matrix $\begin{bmatrix} -2 & -1 & -5 & 1 \\ -1 & -1 & -4 & 0 \\ 0 & 2 & 7 & 3 \\ 1 & 1 & 8 & 4 \end{bmatrix}$ RREF $\rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} a = 2d \\ b = 2d \\ c = -d \end{array}$

$\mathcal{K}(R) = \left\{ \begin{bmatrix} 2d & 2d \\ -d & d \end{bmatrix} \middle| \, d \in \mathbb{C} \right\} = \left\langle \left\{ \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \right\} \right\rangle$

(b) Compute the range of R, $\mathcal{R}(R)$.

Theorem SSRLT: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ spans M_{22}

$\mathcal{R}(R) = \langle \{ -2 - x + 0x^2 + x^3,\ -1 - x + 2x^2 + x^3,\ -5 - 4x + 7x^2 + 8x^3,\ 1 + 3x^2 + 4x^3 \} \rangle$

↑ images of 4 basis vectors.

(c) Is R an invertible linear transformation? Why or why not?

$\mathcal{K}(R) \neq \{\underline{0}\}$ so by Theorem KILT, R is <u>not</u> injective.

By Theorem ILTIS, R is not invertible

4. Consider the invertible linear transformation T from the vector space $\mathbb{C}^3$ to the vector space of polynomials P_2. Compute an explicit formula for the inverse of T, the linear transformation $T^{-1}\colon P_2 \to \mathbb{C}^3$. (20 points)

$$T\colon \mathbb{C}^3 \to P_2, \quad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (-5a - 2b + 2c) + (2a + b - c)x + (-3a - b)x^2$$

Nice basis of codomain, $C = \{1, x, x^2\}$.

<u>Pre-images</u> $T^{-1}(1) \Rightarrow$ system $\left[\begin{array}{ccc|c} -5 & -2 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ -3 & -1 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$T^{-1}(1) = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$ adjust vector of constants, solve two more

$T^{-1}(x) = \left\{ \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix} \right\}$ $T^{-1}(x^2) = \left\{ \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}$

$T^{-1}(a + bx + cx^2) = T^{-1}(a1 + bx + cx^2) = aT^{-1}(1) + bT^{-1}(x) + cT^{-1}(x^2)$

$= a\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -a - 2b \\ 3a + 6b - c \\ a + b - c \end{bmatrix}$

5. Suppose that $T: U \to V$ and $S: V \to W$ are linear transformations. Prove that the composition of S and T, $S \circ T$, is a linear transformation. (15 points)

T is a LT

$$(S \circ T)(\underset{\sim}{x}+\underset{\sim}{y}) = S(T(\underset{\sim}{x}+\underset{\sim}{y})) = S(T(\underset{\sim}{x})+T(\underset{\sim}{y}))$$

$$= S(T(\underset{\sim}{x})) + S(T(\underset{\sim}{y})) = (S \circ T)(\underset{\sim}{x}) + (S \circ T)(\underset{\sim}{y})$$

S is a LT

$$(S \circ T)(\alpha \underset{\sim}{x}) = S(T(\alpha \underset{\sim}{x})) = S(\alpha T(\underset{\sim}{x})) = \alpha S(T(\underset{\sim}{x})) = \alpha (S \circ T)(\underset{\sim}{x})$$

T is LT S is LT

6. Suppose that $T: U \to V$ is a linear transformation which has an inverse **function**, T^{-1}. To prove that T^{-1} is a linear transformation would require checking two defining properties. Choose **one** of the two properties, and prove it. (15 points)

$T^{-1}(\underset{\sim}{v_1}+\underset{\sim}{v_2}) = ?$

$v_1 \in V$, T onto $\Rightarrow u_1 \in U$ $T(\underset{\sim}{u_1}) = \underset{\sim}{v_1}$

$v_2 \in V$, T onto $\Rightarrow u_2 \in U$ $T(\underset{\sim}{u_2}) = \underset{\sim}{v_2}$

$$= T^{-1}(T(\underset{\sim}{u_1})+T(\underset{\sim}{u_2})) = T^{-1}(T(\underset{\sim}{u_1}+\underset{\sim}{u_2}))$$

T is a linear transformation

$$= \underset{\sim}{u_1}+\underset{\sim}{u_2}$$ since T^{-1} is the inverse function

$$= T^{-1}(\underset{\sim}{v_1}) + T^{-1}(\underset{\sim}{v_2})$$

rearrange these