Math 290 Name: Key Dr. Beezer
Exam 7 Fall 2019
Chapter R

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices, compute determinants, and compute eigenvalues, eigenmatrices and eigenspaces. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Let $\mathbf{u}$ be an element of the vector space of 2×2 symmetric matrices, S_{22}, and let B be a basis for the vector space. Compute the vector representation of $\mathbf{u}$ relative to B, $\rho_B(\mathbf{u})$. (15 points)

$$\mathbf{u}=\begin{bmatrix}2 & 4\\ 4 & -3\end{bmatrix}\qquad B=\left\{\begin{bmatrix}1 & -2\\ -2 & 0\end{bmatrix},\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix},\begin{bmatrix}6 & -8\\ -8 & 5\end{bmatrix}\right\}$$

$$\begin{bmatrix}2 & 4\\ 4 & -3\end{bmatrix} = a\begin{bmatrix}1 & -2\\ -2 & 0\end{bmatrix}+b\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+c\begin{bmatrix}6 & -8\\ -8 & 5\end{bmatrix}$$

yields a system of 3 equations w/ 3 variables;

$$\left[\begin{array}{ccc|c}1 & 1 & 6 & 2\\ -2 & -1 & -8 & 4\\ 0 & 1 & 5 & -3\end{array}\right]\xrightarrow{\text{RREF}}\left[\begin{array}{ccc|c}1 & 0 & 0 & 16\\ 0 & 1 & 0 & 52\\ 0 & 0 & 1 & -11\end{array}\right]\text{ so } \rho_B(\mathbf{u})=\begin{bmatrix}16\\ 52\\ -11\end{bmatrix}$$

2. E and F are bases for the vector space $\mathbb{C}^2$. Compute the change-of-basis matrix, $C_{E,F}$. (15 points)

$$E=\left\{\begin{bmatrix}1\\ 3\end{bmatrix},\begin{bmatrix}2\\ -1\end{bmatrix}\right\}\qquad F=\left\{\begin{bmatrix}-1\\ 11\end{bmatrix},\begin{bmatrix}1\\ -18\end{bmatrix}\right\}$$

$$\begin{bmatrix}1\\ 3\end{bmatrix}=-3\begin{bmatrix}-1\\ 11\end{bmatrix}+2\begin{bmatrix}1\\ -18\end{bmatrix}\Rightarrow \rho_F\left(\begin{bmatrix}1\\ 3\end{bmatrix}\right)=\begin{bmatrix}-3\\ -2\end{bmatrix}$$

$$\begin{bmatrix}2\\ -1\end{bmatrix}=-5\begin{bmatrix}-1\\ 11\end{bmatrix}+-3\begin{bmatrix}1\\ -18\end{bmatrix}\Rightarrow \rho_F\left(\begin{bmatrix}2\\ -1\end{bmatrix}\right)=\begin{bmatrix}-5\\ -3\end{bmatrix}$$

$$\Rightarrow C_{E,F}=\begin{bmatrix}-3 & -5\\ -2 & -3\end{bmatrix}$$

linear combinations
from solving systems

3. Consider the linear transformation T from the vector space of 1×2 matrices, M_{12}, to the vector space of polynomials with largest degree 1, P_1. B and E are bases of M_{12}, C and F are bases of P_1. ($\cancel{20}$ points)

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$$T\colon M_{12} \to P_1, \quad T\left(\begin{bmatrix} a & b \end{bmatrix}\right) = (2a + 3b) + (a + b)x$$
$$B = \left\{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\right\} \quad C = \{1, x\}$$
$$E = \left\{\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix}\right\} \quad F = \{2 + x, 3 + x\}$$
$$\mathbf{u} = \begin{bmatrix} 3 & 1 \end{bmatrix}$$

(a) Compute $T(\mathbf{u})$ using the definition above.

$$T([3\ 1]) = (2(3) + 3(1)) + (3+1)x = 9 + 4x$$

(b) Compute the matrix representation of T relative to B and C.

$$\rho_C(T([1\ 0])) = \rho_C(2+x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad M^T_{B,C} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$
$$\rho_C(T([0\ 1])) = \rho_C(3+x) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(c) Compute $T(\mathbf{u})$, using the matrix representation from (b) and two vector representation linear transformations (ρ_X).

$$T(\underset{\sim}{u}) = \rho_C^{-1}\left(M^T_{B,C}\, \rho_B([3\ 1])\right) = \rho_C^{-1}\left(\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \rho_C^{-1}\left(\begin{bmatrix} 9 \\ 4 \end{bmatrix}\right)$$
$$= 9(1) + 4(x) = 9 + 4x$$

(d) Compute the matrix representation of T relative to E and F.

$$\rho_F(T([1\ 1])) = \rho_F(5 + 2x) = \rho_F(1(2+x) + 1(3+x)) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\rho_F(T([3\ 4])) = \rho_F(18 + 7x) = \rho_F(3(2+x) + 4(3+x)) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

two (similar) systems needed

$$M^T_{E,F} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

(e) Compute $T(\mathbf{u})$, using the matrix representation from (d) and two vector representation linear transformations (ρ_X).

$$T(\underset{\sim}{u}) = \rho^{-1}\left(M^T_{E,F}\, \rho_E([3\ 1])\right) = \rho_F^{-1}\left(\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 9 \\ -2 \end{bmatrix}\right)$$

$(9[1\ 1] + -2[3\ 4] = [3\ 1])$

$$= \rho_F^{-1}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = 3(2+x) + 1(3+x) = 9 + 4x \quad (\text{again!})$$

4. Consider the invertible linear transformation T from the vector space $\mathbb{C}^3$ to the vector space of polynomials P_2. Compute an explicit formula for the inverse of T, the linear transformation $T^{-1}\colon P_2 \to \mathbb{C}^3$. Do this using techniques from Chapter R, not with techniques from Chapter LT, such as computing pre-images. Do not skip any steps for dealing with $\mathbb{C}^3$ that might seem trivial, they must be described explicitly. (15 points)

$$T\colon \mathbb{C}^3 \to P_2, \quad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (-5a - 2b + 2c) + (2a + b - c)x + (-3a - b)x^2$$

Choose nice bases $B = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ & $C = \{1, x, x^2\}$

Matrix representation, "on sight"

$$M^T_{B,C} = \begin{bmatrix} -5 & -2 & 2 \\ 2 & 1 & -1 \\ -3 & -1 & 0 \end{bmatrix} \quad \text{then} \quad M^{T^{-1}}_{C,B} = \left(M^T_{B,C}\right)^{-1} = \begin{bmatrix} -1 & -2 & 0 \\ 3 & 6 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\text{So } T^{-1}(a+bx+cx^2) = \rho_B^{-1}\left(M^{T^{-1}}_{C,B}\, \rho_C(a+bx+cx^2)\right)$$

$$= \rho_B^{-1}\left(\begin{bmatrix} -1 & -2 & 0 \\ 3 & 6 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \rho_B^{-1}\left(\begin{bmatrix} -a-2b \\ 3a+6b-c \\ a+b-c \end{bmatrix}\right)$$

like last time!

necessary step

$$= (-a-2b)\begin{bmatrix}1\\0\\0\end{bmatrix} + (3a+6b-c)\begin{bmatrix}0\\1\\0\end{bmatrix} + (a+b-c)\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix} -a-2b \\ 3a+6b-c \\ a+b-c \end{bmatrix}$$

5. Find a basis for P_2, the vector space of polynomials of degree at most 2, which will yield a diagonal matrix representation for the linear transformation R. (15 points)

$$R\colon P_2 \to P_2, \qquad R(a + bx + cx^2) = (-5a + 4b) + (-8a + 7b)x + (4a - 4b - c)x^2$$

Make a matrix representation "on sight" with the nice basis $B = \{1, x, x^2\}$

$$M^R_{B,B} = \begin{bmatrix} -5 & 4 & 0 \\ -8 & 7 & 0 \\ 4 & -4 & -1 \end{bmatrix}$$

A basis of eigenvectors of $M^R_{B,B}$ via Sage

$$C = \left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

Un-coordinatize, via ρ_B^{-1} (linear combinations), to get a basis of P_2 that would yield a diagonal representation (with the eigenvalues $3, -1, -1$ on the diagonal).

$$\{1+2x-x^2,\ 1+x,\ x^2\}$$