Math 290 Name: Key Dr. Beezer
Exam 2 Spring 2019
Chapter V

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Is the vector $\mathbf{x}$ an element of the span of S, $\langle S\rangle$? Explain carefully why, or why not. (15 points)

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} \qquad S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -8 \\ -2 \\ 5 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ -5 \\ 4 \end{bmatrix} \right\}$$

① Defn SS, are there scalars a_1, a_2, a_3 so that $a_1 \underset{\sim}{v_1} + a_2 \underset{\sim}{v_2} + a_3 \underset{\sim}{v_3} = \underset{\sim}{x}$?

② By Theorem SLSLC, solve system w/ augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -8 & 1 & 1 \\ 1 & -2 & -1 & 2 \\ -1 & 5 & -1 & 1 \\ -1 & 8 & -5 & -2 \\ -1 & -2 & 4 & 3 \end{array}\right]$$

③ RREF $\rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

④ Theorem RCLS says the system is inconsistent (last column is a pivot), so no solution, so no linear combination. Thus "no!", or $\underset{\sim}{x} \notin \langle S\rangle$.

2. Is the set T linearly independent? Why or why not? (15 points)

$$T = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3\} = \left\{ \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -5 \\ 0 \\ 7 \end{bmatrix} \right\}$$

LIVRN is most efficient. So make a matrix w/ vectors as columns & row-reduce.

$$[\underset{\sim}{u_1}|\underset{\sim}{u_2}|\underset{\sim}{u_3}] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$n = \#$ columns $= \#$ vectors $= 3$

$r = \#$ pivot columns $= 2$

$2 = r < n = 3$

So by Theorem LIVRN, the set T is linearly dependent.

3. Given the matrix A, use the appropriate theorem to find a linearly independent set R so that the span of R is the null space of A, $\langle R\rangle = \mathcal{N}(A)$. (20 points)

$$A = \begin{bmatrix} -2 & -3 & -8 & -2 & 3 \\ -1 & -2 & -5 & -2 & 2 \\ 1 & 2 & 5 & 2 & -2 \end{bmatrix}$$

Theorem BNS says the vectors we get in the "vector form of the solutions" to homogeneous $LS(A, \mathbf{0})$ will meet requirements.

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3, x_4, x_5 free

$$\begin{bmatrix} \\ \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow$$

Answer:

$$R = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. Use the appropriate theorem to find a set T so that (1) T is a subset of R, (2) T is linearly independent, and (3) the span of T equals the span of R, $\langle T\rangle = \langle R\rangle$. (20 points)

$$R = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -6 \\ -7 \\ -8 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \\ 7 \\ 1 \\ 7 \end{bmatrix} \right\}$$

Theorem BS says to make these vectors the columns of a matrix, row-reduce & identify pivot columns.

$$[\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\mathbf{x}_4|\mathbf{x}_5] \xrightarrow{\text{RREF}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots = $D = \{1, 2, 3, 4\}$

Answer:

$$T = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$$

5. Consider the linear system with coefficient matrix $A = \begin{bmatrix} 2 & 1 & -5 & 8 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ and vector of constants $\mathbf{b} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$. (15 points)

(a) Find a single solution (expressed as a column vector) to $\mathcal{LS}(A, \mathbf{b})$ by setting the free variables to nonzero single-digit integer values (each different). You may use any choice, but make certain your choices fit these requirements.

$[A|b] \xrightarrow[RREF]{} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 4 \\ 0 & 1 & -1 & 2 & -2 \end{array}\right]$ I'll choose $x_3=1,\ x_4=2$

$\Rightarrow x_1=0,\ x_2=-5$

$$\underset{\sim}{x} = \begin{bmatrix} 0 \\ -5 \\ 1 \\ 2 \end{bmatrix}$$

(b) Find a single solution (expressed as a column vector) to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. The free variables will be the same ones as in part (a), and in this part use the *negatives* of the values you used in part (a).

$A \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & 2 \end{bmatrix}$ I must choose $x_3=-1,\ x_4=-2$

Just like above.

$\Rightarrow x_1 = 4,\ x_2=3$ $\quad \underset{\sim}{y} = \begin{bmatrix} 4 \\ 3 \\ -1 \\ -2 \end{bmatrix}$

(c) Theorem PSPHS tells us that the sum of these two solutions will be a solution to $\mathcal{LS}(A, \mathbf{b})$. Compute this sum, and check that it really is a solution.

PSPHS $\Rightarrow \underset{\sim}{x}+\underset{\sim}{y} = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ Solution? $2(4)+1(-2)+-5(0)+8(0)=6$

$-1(4)+1(-2)+1(0)+(-1)(0)=-6$

Yes.

(d) You could have found the solution in part (c) much faster by what procedure? Explain.

$\underset{\sim}{x}+\underset{\sim}{y}$ has free variables $x_3=x_4=0$. The Hayden choice.

Then $x_1=4,\ x_2=-2$, the entries of the last column in part (a).

6. Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Give a proof that $\alpha \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \alpha\mathbf{v} \rangle$. (15 points)

$\alpha\langle \underset{\sim}{u}, \underset{\sim}{v}\rangle = \alpha\left(\overline{[u]_1}[v]_1 + \overline{[u]_2}[v]_2 + \dots + \overline{[u]_m}[v]_m\right)$ Defn IP

$= \overline{[u]_1}\,\alpha[v]_1 + \overline{[u]_2}\,\alpha[v]_2 + \dots + \overline{[u]_m}\,\alpha[v]_m$ Grade 2

$= \overline{[u]_1}[\alpha v]_1 + \overline{[u]_2}[\alpha v]_2 + \dots + \overline{[u]_m}[\alpha v]_m$ Defn CVSM

$= \langle \underset{\sim}{u}, \alpha \underset{\sim}{v}\rangle$ Defn IP