Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices, except in the question that asks you to row-reduce without Sage. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Solve the following two systems of linear equations and express the solutions for each as a set of column vectors. (25 points)

   (a)

   $$3x_1 - 2x_2 - 3x_3 = 2$$
   $$4x_1 - 3x_2 - 7x_3 = 3$$
   $$2x_1 - x_2 \qquad = 2$$
   $$x_1 - 2x_2 - 8x_3 = -3$$

   (b)

   $$-2x_1 + 5x_2 + 8x_3 - 3x_4 = -2$$
   $$x_1 - 3x_2 - 5x_3 + 2x_4 = 1$$
   $$3x_2 + 6x_3 - 3x_4 = 1$$

2. Is the matrix B singular or not? Provide a justification for your answer. (10 points)

$$B = \begin{bmatrix} -1 & 0 & 3 & -1 \\ 1 & -1 & 1 & -5 \\ 1 & 0 & -4 & 2 \\ 1 & -2 & 1 & -7 \end{bmatrix}$$

3. Without using Sage, find a matrix B in reduced row-echelon form which is row-equivalent to A. It is especially important to show all of your work, so it is clear you have not used Sage. (15 points)

$$A = \begin{bmatrix} 1 & 2 & 8 & 3 \\ 1 & 1 & 5 & 1 \\ 2 & 3 & 13 & 4 \end{bmatrix}$$

4. Each part below gives some specific information about a system of linear equations. Say as much as you can about the solution set of the system, with explanations (such as indicating definitions and theorems which apply). (20 points)

   (a) The system is consistent and has more equations than variables.

   (b) The system is homogenous and has more variables than equations.

   (c) The system is homogenous and has the same number of variables as equations.

   (d) The coefficient matrix is nonsingular.

   (e) The system is homogenous and the coefficient matrix is nonsingular.

5. Compute the null space of the matrix B, $\mathcal{N}(B)$. (15 points)

$$B = \begin{bmatrix} 1 & 3 & 3 & -1 \\ -3 & -8 & -8 & 2 \\ -1 & -7 & -7 & 5 \end{bmatrix}$$

6. Consider the following theorem (which you are to *use*, but not *prove*): Suppose that

$$x_1 = a_1, x_2 = a_2, x_3 = a_3, \ldots, x_n = a_n \qquad \text{and} \qquad x_1 = b_1, x_2 = b_2, x_3 = b_3, \ldots, x_n = b_n$$

are two solutions to $\mathcal{LS}(A, \mathbf{b})$. Then a solution to the homogenous system $\mathcal{LS}(A, \mathbf{0})$ is

$$x_1 = a_1 - b_1, x_2 = a_2 - b_2, x_3 = a_3 - b_3, \ldots, x_n = a_n - b_n.$$

Use this theorem and the definition of a nonsingular matrix to prove: If A is a nonsingular matrix, then any solution to $\mathcal{LS}(A, \mathbf{b})$ is unique. (15 points)