Math 290 Name: Key Dr. Beezer
Exam 5 Spring 2020
Chapters D&E

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage as allowed in the statement of a question. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Compute the determinant of A, without using Sage (so be sure to show all your work). (15 points)

$$A = \begin{bmatrix} 0 & 1 & 5 & 5 \\ 3 & 0 & 0 & 2 \\ 2 & 3 & 8 & 8 \\ 1 & 2 & -1 & -1 \end{bmatrix} \leftarrow \text{expand about row 2, multiple zeros}$$

$$= (-1)3 \begin{vmatrix} 1 & 5 & 5 \\ 3 & 8 & 8 \\ 2 & -1 & -1 \end{vmatrix} + 0(\ ) + 0(\ ) + (1)(2) \begin{vmatrix} 0 & 1 & 5 \\ 2 & 3 & 8 \\ 1 & 2 & -1 \end{vmatrix} \leftarrow \text{expand about column 1 w/ a zero}$$

equal columns ⇒ zero determinant

$$= 2\left[0(\ ) + (-1)(2)\begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix} + 1(1)\begin{vmatrix} 1 & 5 \\ 3 & 8 \end{vmatrix}\right] = 2\left((-2)(-11) + (-7)\right)$$

$$= 2 \cdot 15 = 30$$

2. Compute the eigenvalues, eigenspaces, algebraic multiplicities, and geometric multiplicities of B. You may use Sage to obtain a factored characteristic polynomial and to row-reduce matrices. (20 points)

$$B = \begin{bmatrix} 7 & 12 & -36 \\ -8 & -15 & 48 \\ -2 & -4 & 13 \end{bmatrix}$$

$B.\text{fcp}() = (x-3)(x-1)^2$ $\lambda = 3, 1, 1$ eigenvalues

$\alpha_B(3) = 1,\ \alpha_B(1) = 2$

$$B - 3I_3 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathcal{E}_B(3) = \mathcal{N}(B - 3I_3) = \left\langle\left\{\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}\right\}\right\rangle, \ \gamma_B(3) = 1$$

$$B - 1I_3 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathcal{E}_B(1) = \mathcal{N}(B - 1I_3) = \left\langle\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$\gamma_B(1) = 2$

3. Determine if the matrix in each part can be diagonalized. If the matrix cannot be diagonalized, give an explanation demonstrating the application of a theorem. When the matrix can be diagonalized, find a nonsingular matrix and a diagonal matrix so that a similarity transformation by the nonsingular matrix will produce the diagonal matrix. You may use Sage to obtain a factored characteristic polynomial and to row-reduce matrices. (35 points)

(a) $C = \begin{bmatrix} 5 & 3 & 9 & -36 \\ 12 & 14 & -9 & -54 \\ 6 & 6 & -1 & -30 \\ 3 & 3 & 0 & -16 \end{bmatrix}$

$C.\text{fcp}() = (x-2)^2 (x+1)^2$ $\quad \lambda = 2, 2, -1, -1$

$\alpha_C(2) = 2, \ \alpha_C(-1) = 2$

$C - 2I_4 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 0 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ two lin ind eigenvectors $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}$; $\gamma_C(2) = 2$

$C - (-1)I_4 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 & -7 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ two lin. ind. eigenvectors $\begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix}$; $\gamma_C(-1) = 2$

Theorem DMFE says C is diagonalizable & proof of Theorem DC provides S & D

$D = \begin{bmatrix} 2 & & & 0 \\ & 2 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$; $S = \begin{bmatrix} -1 & 6 & -3 & 7 \\ 1 & 0 & 3 & -2 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

(b) $E = \begin{bmatrix} -16 & -54 & 18 & -36 \\ 11 & 31 & -7 & 17 \\ 2 & 2 & 4 & -1 \\ -7 & -25 & 11 & -17 \end{bmatrix}$

$E.\text{fcp}() = (x-2)^2 (x-1)^2$

Eigenvalues $\lambda = 2, 2, -1, -1$ so $\alpha_E(-1) = 2$

$E - (-1)I_4 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -12/5 \\ 0 & 1 & 0 & 7/5 \\ 0 & 0 & 1 & 1/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ nullity 1 so $\gamma_E(-1) = \dim(N(E - (-1)I_4)) = 1$

Since $\alpha_E(-1) = 2 \neq 1 = \gamma_E(-1)$

Theorem DMFE says E is <u>not</u> diagonalizable

4. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of the matrix A for the eigenvalue λ. Suppose $\alpha, \beta \in \mathbb{C}$ are such that $\alpha\mathbf{x} + \beta\mathbf{y} \neq \mathbf{0}$. Prove that then $\alpha\mathbf{x} + \beta\mathbf{y}$ is an eigenvector of A. (15 points)

Check the matrix-vector product:

$$A(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y}) = A(\alpha \underset{\sim}{x}) + A(\beta \underset{\sim}{y}) \qquad \text{MMDAA}$$
$$= \alpha (A\underset{\sim}{x}) + \beta (A\underset{\sim}{y})$$
$$= \alpha(\lambda \underset{\sim}{x}) + \beta(\lambda \underset{\sim}{y}) \qquad \text{Hypothesis}$$
$$= \lambda(\alpha \underset{\sim}{x}) + \lambda(\beta \underset{\sim}{y})$$
$$= \lambda(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y})$$

So $\alpha \underset{\sim}{x} + \beta \underset{\sim}{y}$ is an eigenvector of A (for λ).

5. Suppose that A is a matrix similar to $B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Determine, with proof, a matrix similar to A^5. (15 points)

B^5. Why?

A & B similar $\Rightarrow$ there exists S so that $A = S^{-1}BS$.

So $A^5 = (S^{-1}BS)^5 = S^{-1}BSS^{-1}BSS^{-1}BSS^{-1}BSS^{-1}BS$
$= S^{-1}BIBIBIBIBS$
$= S^{-1}B^5S$ So A^5 & B^5 are similar,

AND $B^5 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}^5 = \begin{bmatrix} (-2)^5 & & 0 \\ & 1^5 & \\ 0 & & 3^5 \end{bmatrix} = \begin{bmatrix} -32 & & 0 \\ & 1 & \\ 0 & & 243 \end{bmatrix}$