Math 290 Name: Dr. Beezer
Exam 6 Spring 2020
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features). P_n is the vector space of polynomials with degree n or less, and $M_{m,n}$ is the vector space of $m \times n$ matrices.

1. Use the *definition* of a linear transformation to verify that T is a linear transformation. (15 points)

$$T\colon \mathbb{C}^2 \to \mathbb{C}^2 \qquad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a + b \\ a - b \end{bmatrix}$$

2. Find a basis for the range of T, $\mathcal{R}(T)$. (20 points)

$$T\colon \mathbb{C}^3 \to P_2 \qquad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a - 2b - c) + (2a + b + 3c)x + (3a + 3c)x^2$$

3. Consider the linear transformation S. (35 points)

$S\colon M_{1,3} \to P_2 \qquad S\left(\begin{bmatrix} a & b & c \end{bmatrix}\right) = (2a + 7b + 6c) + (a + 4b + 4c)x + (-b - c)x^2$

(a) Demonstrate that S is invertible.

(b) Find a formula for $S^{-1}\colon P_2 \to M_{1,3}$.

(c) Compute the composition $S^{-1} \circ S$ and explain why this is a partial check on your work above.

4. The linear transformation R is not injective (you may assume this). Find two different vectors, $\mathbf{x}$ and $\mathbf{y}$, such that $R(\mathbf{x}) = R(\mathbf{y})$. (15 points)

$$R\colon \mathbb{C}^4 \to P_3 \qquad R\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (b + c - 2d) + (-a + 3c + 5d)x + (2b + 3c - 3d)x^2 + (4b + 8c - 4d)x^3$$

5. Suppose that $T\colon U \to V$ and $S\colon U \to V$ are linear transformations which agree on a basis of U. That is, for some basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ of U, $T(\mathbf{u}_i) = S(\mathbf{u}_i)$ for $1 \leq i \leq n$. Prove that T and S are equal functions. (15 points)