

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.

Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices, compute determinants of matrices, and compute eigenstuff of matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features). P_n is the vector space of polynomials with degree n or less, and $M_{m,n}$ is the vector space of $m \times n$ matrices.

1. Find the matrix representation of the linear transformation below, relative to the provided bases. (15 points)

$$T: \mathbb{C}^2 \rightarrow P_1, \quad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (a + 3b) + (2a + b)x$$

$$B = \left\{ \begin{bmatrix} -7 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}, \quad C = \{5 + 4x, 9 + 7x\}$$

2. Use a matrix representation to find the inverse of the invertible linear transformation R given below. (You may assume the linear transformation is invertible, and you must use a matrix representation in your solution to receive any credit.) (20 points)

$$R: P_2 \rightarrow \mathbb{C}^3 \quad R(a + bx + cx^2) = \begin{bmatrix} 5b + c \\ -2a + 7b + 2c \\ -a + 3b + c \end{bmatrix}$$



3. Find a basis for the range of the linear transformation S defined below. (20 points)

$$S: M_{22} \rightarrow P_2 \quad S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (-3a + 2b - 2d) + (a - b + 4c - d)x + (-a + 8c - 4d)x^2$$

4. Find eigenvalues and eigenspaces of the linear transformation S defined below. (20 points)

$$S: P_2 \rightarrow P_2 \quad S(a + bx + cx^2) = (16a - 15b + 10c) + (10a - 9b + 10c)x + (-10a + 15b - 4c)x^2$$



5. Consider the linear transformation T defined below and with bases B and C for P_2 , D and E for \mathbb{C}^2 . (25 points)

$$T: P_2 \rightarrow \mathbb{C}^2 \quad T(a + bx + cx^2) = \begin{bmatrix} a - b + c \\ 2a + 3c \end{bmatrix}$$

$$B = \{-1 - x + 2x^2, -1 - x + 3x^2, 1 + 2x - 8x^2\} \quad C = \{2 + 7x^2, -3 - x - 5x^2, 2 - x + 12x^2\}$$

$$D = \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix} \right\} \quad E = \left\{ \begin{bmatrix} -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$$

- (a) Construct the two change-of-basis matrices: first for converting vector representations from B to C , and then for converting vector representations from D to E .

- (b) Build two matrix representations of T : first with bases B and D and then with bases C and E .

- (c) Write a general expression, using proper notation, that relates the four matrices in the previous two parts.

- (d) Collectively verify your answers above by demonstrating that your four matrices obey the relationship given in part (c).

