Math 290 Name: Key Dr. Beezer
Exam 7 Spring 2020
Chapter R

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices, compute determinants of matrices, and compute eigenstuff of matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features). $P_n$ is the vector space of polynomials with degree $n$ or less, and $M_{m,n}$ is the vector space of $m \times n$ matrices.

1. Find the matrix representation of the linear transformation below, relative to the provided bases. (15 points)

$$T\colon \mathbb{C}^2 \to P_1, \quad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (a+3b) + (2a+b)x$$

$$B = \left\{\begin{bmatrix} -7 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}, \quad C = \{5+4x,\, 9+7x\}$$

$$\rho_C\left(T\left(\begin{bmatrix} -7 \\ 2 \end{bmatrix}\right)\right) = \rho_C(-1-12x) = \rho_C\left(-101(5+4x)+56(9+7x)\right) = \begin{bmatrix} -101 \\ 56 \end{bmatrix}$$

$$\rho_C\left(T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)\right) = \rho_C(6+7x) = \rho_C\left(21(5+4x)+(-11)(9+7x)\right) = \begin{bmatrix} 21 \\ -11 \end{bmatrix}$$

So $M^T_{B,C} = \begin{bmatrix} -101 & 21 \\ 56 & -11 \end{bmatrix}$

2. Use a matrix representation to find the inverse of the invertible linear transformation $R$ given below. (You may assume the linear transformation is invertible, and you must use a matrix representation in your solution to receive any credit.) (20 points)

$$R\colon P_2 \to \mathbb{C}^3 \qquad R\left(a+bx+cx^2\right) = \begin{bmatrix} 5b+c \\ -2a+7b+2c \\ -a+3b+c \end{bmatrix}$$

On sight w/ bases $B = \{1, x, x^2\}$, $C = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$;

$$M^R_{B,C} = \begin{bmatrix} 0 & 5 & 1 \\ -2 & 7 & 2 \\ -1 & 3 & 1 \end{bmatrix} \quad \text{So} \quad M^{R^{-1}}_{C,B} = \left(M^R_{B,C}\right)^{-1} = \begin{bmatrix} & \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 1 & -5 & 10 \end{bmatrix}$$

$$R^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \rho_B^{-1}\left(M^{R^{-1}}_{B,C}\,\rho_C\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)\right) = \rho_B^{-1}\left(\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 1 & -5 & 10 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \rho_B^{-1}\left(\begin{bmatrix} a-2b+3c \\ b-2c \\ a-5b+10c \end{bmatrix}\right)$$

$$= (a-2b+3c)(1) + (b-2c)x + (a-5b+10c)x^2$$

3. Find a basis for the range of the linear transformation $S$ defined below. (20 points)

$$S\colon M_{22} \to P_2 \qquad S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (-3a + 2b - 2d) + (a - b + 4c - d)x + (-a + 8c - 4d)x^2$$

Matrix representation, on sight, w/ bases $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$C = \{1, x, x^2\}$

$$M^S_{B,C} = \begin{bmatrix} -3 & 2 & 0 & -2 \\ 1 & -1 & 4 & -1 \\ -1 & 0 & 8 & -4 \end{bmatrix} \qquad \mathcal{R}(S) \cong \mathcal{C}(M^S_{B,C})$$

$$\xleftarrow{\rho_C^{-1}}$$

basis

Column space as row space of $M^T$, row reduce

$$\mathcal{C}(M^S_{B,C}) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\} \right\rangle$$

$$M^T \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho_C^{-1} \Downarrow$ coordinatization principle

$$\mathcal{R}(S) = \left\langle \left\{ 1 + x^2,\ x + 2x^2 \right\} \right\rangle$$

basis

4. Find eigenvalues and eigenspaces of the linear transformation $S$ defined below. (20 points)

$$S\colon P_2 \to P_2 \qquad S\left(a + bx + cx^2\right) = (16a - 15b + 10c) + (10a - 9b + 10c)x + (-10a + 15b - 4c)x^2$$

Basis $B = \{1, x, x^2\}$, on sight

$$M^S_{B,B} = \begin{bmatrix} 16 & -15 & 10 \\ 10 & -9 & 10 \\ -10 & 15 & -4 \end{bmatrix}$$

Sage: .eigenmatrix_right() output:
$$\begin{bmatrix} -9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & 3/2 \end{bmatrix}$$

un-coordinatized relative to $B$

eigenvalue $\lambda = -9$ $\qquad \mathcal{E}_S(-9) = \left\langle \{ 1 + x - x^2 \} \right\rangle$

eigenvalue $\lambda = 6$ $\qquad \mathcal{E}_S(6) = \left\langle \{ 1 - x^2,\ 2x + 3x^2 \} \right\rangle$

twice what Sage provides to avoid fractions

5. Consider the linear transformation $T$ defined below and with bases $B$ and $C$ for $P_2$, $D$ and $E$ for $\mathbb{C}^2$. (25 points)

$$T\colon P_2 \to \mathbb{C}^2 \qquad T\left(a+bx+cx^2\right) = \begin{bmatrix} a-b+c \\ 2a+3c \end{bmatrix}$$

$$B = \left\{-1-x+2x^2,\ -1-x+3x^2,\ 1+2x-8x^2\right\} \qquad C = \left\{2+7x^2,\ -3-x-5x^2,\ 2-x+12x^2\right\}$$

$$D = \left\{\begin{bmatrix}-1\\-1\end{bmatrix}, \begin{bmatrix}5\\4\end{bmatrix}\right\} \qquad E = \left\{\begin{bmatrix}-3\\-2\end{bmatrix}, \begin{bmatrix}5\\3\end{bmatrix}\right\}$$

(a) Construct the two change-of-basis matrices: first for converting vector representations from $B$ to $C$, and then for converting vector representations from $D$ to $E$.

$$\rho_C(-1-x+2x^2) = \rho_C(1(2+7x^2)+1(-3-x-5x^2)+0(2-x+12x^2)) = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\rho_C(-1-x+3x^2) = \rho_C(6(2+7x^2)+3(-3-x-5x^2)+-2(2-x+12x^2)) = \begin{bmatrix}6\\3\\-2\end{bmatrix}$$

$$\rho_C(1+2x-8x^2) = \rho_C(-5(2+7x^2)+-3(-3-x-5x^2)+1(2-x+12x^2)) = \begin{bmatrix}-5\\-3\\1\end{bmatrix}$$

$$\Rightarrow C_{B,C} = \begin{bmatrix}1 & 6 & -5\\ 1 & 3 & -3\\ 0 & -2 & 1\end{bmatrix}$$

$$\rho_E\left(\begin{bmatrix}-1\\-1\end{bmatrix}\right) = \rho_E\left(2\begin{bmatrix}-3\\-2\end{bmatrix}+1\begin{bmatrix}5\\3\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\rho_E\left(\begin{bmatrix}5\\4\end{bmatrix}\right) = \rho_E\left(-5\begin{bmatrix}-3\\-2\end{bmatrix}+-2\begin{bmatrix}5\\3\end{bmatrix}\right) = \begin{bmatrix}-5\\-2\end{bmatrix}$$

$$\Rightarrow C_{D,E} = \begin{bmatrix}2 & -5\\ 1 & -2\end{bmatrix}$$

(b) Build two matrix representations of $T$: first with bases $B$ and $D$ and then with bases $C$ and $E$.

$$\rho_D(T(-1-x+2x^2)) = \rho_D\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = \rho_D\left(-12\begin{bmatrix}-1\\-1\end{bmatrix}+-2\begin{bmatrix}5\\4\end{bmatrix}\right) = \begin{bmatrix}-12\\-2\end{bmatrix}$$

$$\rho_D(T(-1-x+3x^2)) = \rho_D\left(\begin{bmatrix}3\\7\end{bmatrix}\right) = \rho_D\left(-23\begin{bmatrix}-1\\-1\end{bmatrix}+-4\begin{bmatrix}5\\4\end{bmatrix}\right) = \begin{bmatrix}-23\\-4\end{bmatrix}$$

$$\rho_D(T(1+2x-8x^2)) = \rho_D\left(\begin{bmatrix}-9\\-22\end{bmatrix}\right) = \rho_D\left(74\begin{bmatrix}-1\\-1\end{bmatrix}+13\begin{bmatrix}5\\4\end{bmatrix}\right) = \begin{bmatrix}74\\13\end{bmatrix}$$

$$M^T_{B,D} = \begin{bmatrix}-12 & -23 & 74\\ -2 & -4 & 13\end{bmatrix}$$

$$\rho_E(T(2+7x^2)) = \rho_E\left(\begin{bmatrix}9\\25\end{bmatrix}\right) = \rho_E\left(-98\begin{bmatrix}-3\\-2\end{bmatrix}+-57\begin{bmatrix}5\\4\end{bmatrix}\right) = \begin{bmatrix}-98\\-57\end{bmatrix}$$

(c) Write a general expression, using proper notation, that relates the four matrices in the previous two parts.

$$M^T_{C,E} = C_{D,E}\, M^T_{B,D}\, C_{C,B} = C_{D,E}\, M^T_{B,D}\, C^{-1}_{B,C}$$

$$\rho_E(T(-3-x-5x^2)) = \rho_E\left(\begin{bmatrix}-7\\-21\end{bmatrix}\right) = \rho_E\left(84\begin{bmatrix}-3\\-2\end{bmatrix}+49\begin{bmatrix}5\\4\end{bmatrix}\right) = \begin{bmatrix}84\\49\end{bmatrix}$$

$$\rho_E(T(2-x+12x^2)) = \rho_E\left(\begin{bmatrix}15\\40\end{bmatrix}\right) = \rho_E\left(-155\begin{bmatrix}-3\\-2\end{bmatrix}+-90\begin{bmatrix}5\\4\end{bmatrix}\right) = \begin{bmatrix}-155\\-90\end{bmatrix}$$

(d) Collectively verify your answers above by demonstrating that your four matrices obey the relationship given in part (c).

$$\text{so } M^T_{C,E} = \begin{bmatrix}-98 & 84 & -155\\ -57 & 49 & -90\end{bmatrix}$$