Math 491 Monday, March 30 Chapter 21 Fields

Mon - 21
Tue - 21
Wed - Exam 19/20 8:30 Pacific
Thu - 21
Fri - 21
Mon - Problems <u>Sage 21</u>

$\alpha \in E$ <u>algebraic</u> element over F if α is a root of a poly. in $F[X]$

<u>Theorem</u> α algebraic in E over F. $\Rightarrow$ there exists <u>monic</u> <u>irreducible</u> $p(x) \in F[X]$ of <u>smallest degree</u> with <u>α as a root</u>. Any other monic polynomial in $F[X]$ w/ α as root is <u>divisible</u> by $p(x)$.

<u>Defn</u> $p(x)$ is <u>minimal</u> <u>polynomial</u> of α.

Proof $\phi_\alpha(x) : F[x] \to E$, $\phi_\alpha(p(x)) = \underline{p(\alpha)}$ evaluation homomorphism

$\alpha \in E$, coeffs $\in E$

$\underline{\mathrm{Ker}(\phi_\alpha)}$?

① Ideal in $F[x]$

② $F[x]$ PID

③ $\Rightarrow$ $\mathrm{Ker}(\phi_\alpha) = \langle p(x) \rangle$

generator of principal ideal
this will be minimal polynomial

Grab __any__ poly w/ α as a root, $q(x) \in F[x]$. Then $q(x) \in \mathrm{Ker}(\phi_\alpha)$

So $q(x)$ in a multiple of $p(x)$.

$\Rightarrow$ So $p(x)$ has degree less than or equal to degree of $q(x)$

WLOG scale $p(x)$ by a constant to make/assume $p(x)$ is monic

This makes $p(x)$ unique

Suppose $p(x)$ reducible: $\underline{p(x) = r(x)s(x)}$ $\Rightarrow$ $0 = p(\alpha) = r(\alpha)s(\alpha)$ $\not\Leftarrow\!\!\!\Rightarrow$

$\Rightarrow$ $r(\alpha) = 0$ or $s(\alpha) = 0$ & r & s have lesser degree

(no zero divisors) So $p(x)$ irreducible

Defn The <u>degree</u> of α is the degree of its minimal polynomial.

Ex $\sqrt{2}+3\sqrt{3}$ is algebraic in $\mathbb{Q}(\sqrt{2},\sqrt{3})$ over $\mathbb{Q}$

$\alpha = \sqrt{2}+3\sqrt{3} \rightarrow \alpha - \sqrt{2} = 3\sqrt{3} \rightarrow (\alpha-\sqrt{2})^2 = (3\sqrt{3})^2 \rightarrow \alpha^2 - 2\sqrt{2}\alpha + 2 = 27$

$\alpha^2 - 25 = 2\sqrt{2}\alpha \rightarrow (\alpha^2-25)^2 = (2\sqrt{2}\alpha)^2 \rightarrow \alpha^4 - 50\alpha^2 + 625 = 8\alpha^2$

$\alpha^4 - 58\alpha^2 + 625 = 0$ so α satisfies $p(x) = x^4 - 58x^2 + 625$

This is the minimal polynomial. Check that $p(x)$ is irreducible.

(Quadratic in x^2 : $p(x) = (x^2)^2 - 58x^2 + 625$, $s(t) = t^2 - 58t + 625$)

$F[x]$

<u>Sage</u>

Defn Suppose α algebraic in E over F. Then $F(\alpha)$ is the smallest field containing F & α.

Theorem α algebraic in E over F w/ minimal polynomial $p(x)$.

Then $F(\alpha) \cong F[x]/\langle p(x) \rangle$.

Proof $\phi_\alpha : F[x] \to E$, $\ker \phi_\alpha = \langle p(x) \rangle$

$\operatorname{Im} \phi_\alpha$ subfield of E

(1) contains ~~copy of~~ F : $\{ \phi_\alpha(a) \mid a \in F \} \subseteq \operatorname{Im}(\phi_\alpha)$

polynomial : $q(x) = a$

$\phi_\alpha(q(x)) = q(\alpha) = a$

(2) contains α

$s(x) = x$ $\qquad \phi_\alpha(s(x)) = s(\alpha) = \alpha \in \operatorname{Im} \phi_\alpha$

So $F[x]/\langle p(x) \rangle$ is a field containing α & F

$F(\alpha)$ is the smallest field containing α & F

$\Rightarrow$ equal ?