Math 491, Friday, April 24 Chapter 23

Mon - Chapter 23 $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$

Tue - Sage

Thu - Chapter 23 (Quintic)

Fri - Problem Session / Sage 23
House keeping

Mon - Exam 4
Chap 22, 23

Tue - Projects (x3)

Final Exam / Tuesday 10AM Pacific

Theorem E finite separable extension of F. Then there is a primitive element $a \in E$ so that $E = F(a)$.

Proof ① E finite $\wedge$ set $\Rightarrow E^* = \langle a \rangle$ so $E = F(a)$

② Assume $E = F(a,b) = [F(a)](b)$ (induction for more)

③ $f(x)$ & $g(x)$ are the minimal polynomials of a & b (respectively)
Let K be the field where they split.
Denote these root of $f(x)$ & $g(x)$ as $a_1, a_2, \dots a_n$; $b = b_1, b_2, \dots, b_m$
Each root has multiplicity 1.

④ Form set $S = \left\{ \frac{a_i - a}{b - b_j} \mid 1 \le i \le n, 2 \le j \le m \right\}$ (no zero denominator)

⑤ Grab $t \notin S$. Set $g = a + tb$.
$t \in F$ (F infinite)

Claim $E = F(g)$

① In $F(g)[x]$ define $h(x) = f(g - tx)$; Then

$$h(b) = f(g - tb) = f(a + tb - tb) = f(a) = 0$$

(f min poly of a)

② Suppose $g = a_i + tb_j$, $j > 1$. Then $a + tb = a_i + tb_j \rightarrow tb - tb_j = a_i - a$

$$t = \frac{a_i - a}{b - b_j}$$

So $g \neq a_i + tb_j$ ⇒⇐

③ Suppose $h(b_j) = 0$, $j > 1$. Then $f(g - tb_j) = 0 \Rightarrow g - tb_j = a_i \Rightarrow g = tb_j + a_i$ ⇒⇐

So $h(b_j) \neq 0$

④ So $h(x)$ & $g(x)$ share exactly one root ($b_1 = b$).

So $h(x)$ & $g(x)$ have only one linear factor in common

⑤ the minimal polynomial of b ($g(x)$) must be a factor of any polynomial with b as a root (like, say, $h(x)$).

So $g(x)$ is linear, $g(x) = x - b$.

⑥ So $b \in F(g)$

$g = a + tb \rightarrow a = g - tb \Rightarrow a \in F(g)$

$\in F(g)$ $\in F$ $b \in F(g)$

$F(a,b) \subseteq F(g)$

$(F(g) \subseteq F(a,b))$ $\Rightarrow F(g) = F(a,b)$