Math 290 Name: Key Dr. Beezer
Exam 2 Spring 2021
Chapter V

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Is the set R linearly independent or not? Give complete reasons for why your answer is correct. (15 points)

$$R = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ 4 \\ 2 \end{bmatrix} \right\}$$

Theorem LIVRN suggests making a matrix w/ the vectors of R as the columns

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ -2 & 1 & 1 \\ -5 & 2 & 4 \\ 2 & -2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r = 2 < 3 = n$

So LIVRN says R is <u>not</u> linearly independent.

2. Is $\mathbf{u}$ in the span of U? In other words is $\mathbf{u} \in \langle U \rangle$? Give complete reasons for why your answer is correct. (20 points)

$$\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad U = \left\{ \begin{bmatrix} -1 \\ 2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix} \right\}$$

$= \{\underset{\sim}{u_1}, \underset{\sim}{u_2}, \underset{\sim}{u_3}\}$

Are there scalars a_1, a_2, a_3 so that $a_1 \underset{\sim}{u_1} + a_2 \underset{\sim}{u_2} + a_3 \underset{\sim}{u_3} = \underset{\sim}{u}$? (Defn of span)

Is there a solution to $LS\left(\begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 3 \\ -2 & 1 & -2 \\ -2 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \\ 1 \end{bmatrix}\right)$? (Application of Theorem SLSLC.)

Augmented matrix:

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 5 \\ 2 & -1 & 3 & 1 \\ -2 & 1 & -2 & 2 \\ -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Last column is a pivot column, so RCLS tells us the system is not consistent. So, no, there are no scalars, and thus $\underset{\sim}{u} \notin \langle U \rangle$

3. Determine a set S that is linear independent, and whose span is the null space of A. That is, $\mathcal{N}(A) = \langle S \rangle$. Give complete reasons for why your answer is correct. (15 points)

$$A = \begin{bmatrix} 1 & 0 & -1 & -5 & 1 \\ -2 & 1 & 0 & 6 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 6 & -4 \end{bmatrix}$$

An application of Theorem BNS will provide exactly the set we desire

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Two non-pivots ($F = \{4, 5\}$) implies two vectors.

$$S = \left\{ \begin{bmatrix} 2 \\ -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The pattern of zeros and ones provides the linear independence. But you don't need to say this, Theorem BNS already does.

4. Determine a set T that is linear independent, and whose span is equal to the span of S. That is, $\langle T \rangle = \langle S \rangle$. Give complete reasons for why your answer is correct. (15 points)

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix} \right\}$$

Theorem BS suggests a matrix whose columns are vectors of S

$$A = \begin{bmatrix} 1 & -5 & 2 & -4 \\ 1 & -4 & 1 & -4 \\ 0 & 3 & -3 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$D = \{1, 2, 4\}$

Use vectors 1, 2 & 4 from the set S.

$$T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix} \right\}$$

is (a) linearly independent

(b) $\langle T \rangle = \langle S \rangle$

<u>all</u> by Theorem BS.

5. Prove that for any $\mathbf{u} \in \mathbb{C}^m$, $1\mathbf{u} = \mathbf{u}$. (15 points)

This is a vector equality.

For $1 \le i \le m$

$$[1\,\underset{\sim}{u}]_i = 1[\underset{\sim}{u}]_i \qquad \text{Scalar multiplication of a vector}$$

$$= [\underset{\sim}{u}]_i \qquad 2^{nd} \text{ Grade multiplication by } 1$$

So vectors $1\underset{\sim}{u}$ & $\underset{\sim}{u}$ have equal entries, so Definition CVE implies that $1\underset{\sim}{u} = \underset{\sim}{u}$.

6. Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ are orthogonal vectors with equal norms. Prove that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal vectors. (20 points)

Check the inner product.

$$\begin{aligned}
\langle \underset{\sim}{u}+\underset{\sim}{v}, \underset{\sim}{u}-\underset{\sim}{v} \rangle &= \langle \underset{\sim}{u}+\underset{\sim}{v}, \underset{\sim}{u} \rangle + \langle \underset{\sim}{u}+\underset{\sim}{v}, -\underset{\sim}{v} \rangle \\
&= \langle \underset{\sim}{u}, \underset{\sim}{u} \rangle + \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle + \langle \underset{\sim}{u}, -\underset{\sim}{v} \rangle + \langle \underset{\sim}{v}, -\underset{\sim}{v} \rangle \\
&= \langle \underset{\sim}{u}, \underset{\sim}{u} \rangle + \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle + (-\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle) + (-\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle) \\
&= \|\underset{\sim}{u}\|^2 + 0 + (-0) + -(\|\underset{\sim}{v}\|^2) \\
&= \|\underset{\sim}{u}\|^2 - \|\underset{\sim}{v}\|^2 = 0
\end{aligned}$$

So $\underset{\sim}{u}+\underset{\sim}{v}$ & $\underset{\sim}{u}-\underset{\sim}{v}$ are orthogonal vectors.