Math 290 Name: Key Dr. Beezer
Exam 3 Spring 2021
Chapter M

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Is the matrix B unitary? Why or why not? (15 points)

$$B = \begin{bmatrix} 2 & 1 & -26 \\ 3 & 2 & 17 \\ 1 & -8 & 1 \end{bmatrix}$$

Are the columns an orthonormal set? (Almost.)

Compute $B^* B = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 69 & 0 \\ 0 & 0 & 966 \end{bmatrix} \neq I_3$. So no.

2. Find the solution set of the linear system $\mathcal{LS}(A,\,\mathbf{b})$ using the inverse of the coefficient matrix. No credit will be given for solutions obtained by other methods. (15 points)

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -3 & 1 & 5 & -8 \\ 2 & 0 & -3 & 4 \\ -2 & 1 & 3 & -3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ -5 \\ 8 \end{bmatrix}$$

$$[A|I_4] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -7 & -4 & -5 & 4 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -6 & -4 & -5 & 4 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 \end{array}\right]$$

NMRRI ⇒ nonsingular
NMUS ⇒ unique solution

CINM ⇒ A^{-1}

Theorem SNCM ⇒
Solution is $A^{-1}\underset{\sim}{b} = \begin{bmatrix} -2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$

"solution set" $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$

3. Consider the matrix A. (40 points)

$$A = \begin{bmatrix} 3 & 14 & -16 & 41 & 60 & 23 \\ 1 & 5 & -6 & 15 & 22 & 9 \\ -1 & -7 & 11 & -25 & -36 & -20 \\ -5 & -22 & 28 & -71 & -100 & -45 \\ 1 & 6 & -9 & 21 & 30 & 16 \end{bmatrix}$$

$$[A \mid I_5] \xrightarrow{\text{RREF}} \left[\begin{array}{cccccc|ccccc} 1 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 30 & 3 & 46 \\ 0 & 1 & 0 & 0 & 2 & -2 & 0 & 0 & -17 & -2 & -27 \\ 0 & 0 & 1 & -2 & -2 & 3 & 0 & 0 & -8 & -1 & -13 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 20 & 3 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 1 & 11 \end{array}\right]$$

(a) Find a linearly independent set S, whose span is the column space of A, $\langle S \rangle = \mathcal{C}(A)$, and whose elements are each a column of A.

$D = \{1, 2, 3\}$ Theorem BCS $\Rightarrow$ Columns 1, 2 & 3 of A

$$S = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 14 \\ 5 \\ -7 \\ -22 \\ 6 \end{bmatrix}, \begin{bmatrix} -16 \\ -6 \\ 11 \\ 28 \\ -9 \end{bmatrix} \right\}$$

(b) Find a linearly independent set T, whose span is the column space of A, $\langle T \rangle = \mathcal{C}(A)$, by using the matrix L from the extended echelon form of A.

$C(A) = N(L)$

$\uparrow$

Theorem FS

$$L = \begin{bmatrix} 1 & 0 & 20 & 3 & 32 \\ 0 & 1 & 7 & 1 & 11 \end{bmatrix}$$

$F = \{3, 4, 5\}$

Theorem BNS $\nearrow$

$$T = \left\{ \begin{bmatrix} -20 \\ -7 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -32 \\ -11 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) Find a linearly independent set R, whose span is the column space of A, $\langle R \rangle = \mathcal{C}(A)$, by using theorems about the row space of a matrix.

$C(A) = R(A^t)$

Theorem CSRST

$$A^t \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -11 & 1 \\ 0 & 1 & 0 & 32 & -3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -11 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 32 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right\}$$

$\leftarrow$ nonzero rows as column vectors

Theorem BRS

(d) Find a linearly independent set U, whose span is the row space of A, $\langle U \rangle = \mathcal{R}(A)$.

$R(A) = R(C)$

$\nearrow$

First 3 rows in left side of extended echelon form above

$$U = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ -2 \\ 3 \end{bmatrix} \right\}$$

nonzero rows of C, written as column vectors

Theorem BRS

(e) Construct a nonzero vector $\mathbf{b}$ from **one** of the sets S, T, R, U (your choice, but say which you are using) and explain how you know that $\mathcal{LS}(A, \mathbf{b})$ has a solution (without simply solving the system).

By Theorem CSCS, <u>any</u> vector from $C(A)$ will create a consistent system (w/ a solution). Vectors from U have the wrong size, just for starters. So <u>one</u> correct answer is

$$\begin{bmatrix} 3 \\ 1 \\ -1 \\ -5 \\ 1 \end{bmatrix} \in S$$

(can you predict the solution for this choice?)

4. Suppose that A is an $m \times n$ matrix, and $\mathcal{O}_{n\times p}$ and $\mathcal{O}_{m\times p}$ are zero matrices of the indicated sizes. Give a careful proof that $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$. (15 points)

This is a matrix equality, so appeal to Definition ME.

For $1 \leq i \leq m$, $1 \leq j \leq p$

$$
\begin{aligned}
[A\mathcal{O}]_{ij} &= \sum_{k=1}^{n} [A]_{ik}[\mathcal{O}]_{kj} \\
&= \sum_{k=1}^{n} [A]_{ik}\, 0 \\
&= \sum_{k=1}^{n} 0 \\
&= 0 \\
&= [\mathcal{O}]_{ij}
\end{aligned}
$$

So by Definition ME, $A\mathcal{O} = \mathcal{O}$

5. Suppose that A is a nonsingular matrix. Prove that $\mathcal{LS}(A, \mathbf{b})$ has a unique solution by first assuming there are two solutions (Proof Technique U), and also using a representation of the system with a matrix-vector product (Theorem SLEMM). Full-credit requires following these suggestions, so in particular, do not simply quote existing therorems to provide a simple one-line proof. (15 points)

Suppose $\mathbf{x}_1$ & $\mathbf{x}_2$ are two solutions. (Proof Technique U)

Then $A\mathbf{x}_1 = \mathbf{b}$ & $A\mathbf{x}_2 = \mathbf{b}$. (Theorem SLEMM)

$A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

Then Definition NM says $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$ (the essence of nonsingular)

So $\mathbf{x}_1 = \mathbf{x}_2$ and the two solutions are one.