Math 290 | Name: Key | Dr. Beezer
Exam 6 | | Spring 2021
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers, unless explicitly suggested in the problem's statement. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Consider the function $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ below. (35 points)

$$T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{bmatrix}2a-b\\a+3b\end{bmatrix}$$

(a) Prove that $T$ is a linear transformation.

By the definition,

$$T(\underset{\sim}{x}+\underset{\sim}{y}) = T\left(\begin{bmatrix}a_1\\b_1\end{bmatrix}+\begin{bmatrix}a_2\\b_2\end{bmatrix}\right) = T\left(\begin{bmatrix}a_1+a_2\\b_1+b_2\end{bmatrix}\right) = \begin{bmatrix}2(a_1+a_2)-(b_1+b_2)\\a_1+a_2+3(b_1+b_2)\end{bmatrix}$$

$$= \begin{bmatrix}(2a_1-b_1)+(2a_2-b_2)\\(a_1+3b_1)+(a_2+3b_2)\end{bmatrix} = \begin{bmatrix}2a_1-b_1\\a_1+3b_1\end{bmatrix}+\begin{bmatrix}2a_2-b_2\\a_2+3b_2\end{bmatrix} = T(\underset{\sim}{x})+T(\underset{\sim}{y})$$

$$T(\alpha\underset{\sim}{x}) = T\left(\alpha\begin{bmatrix}a\\b\end{bmatrix}\right) = T\left(\begin{bmatrix}\alpha a\\\alpha b\end{bmatrix}\right) = \begin{bmatrix}2(\alpha a)-\alpha b\\\alpha a+3(\alpha b)\end{bmatrix} = \begin{bmatrix}\alpha(2a-b)\\\alpha(a+3b)\end{bmatrix} = \alpha\begin{bmatrix}2a-b\\a+3b\end{bmatrix} = \alpha T(\underset{\sim}{x})$$

<u>OR</u>

$$T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{bmatrix}2a-b\\a+3b\end{bmatrix} = \begin{bmatrix}2&-1\\1&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

is a linear transformation by Theorem MBLT.

(b) Is $T$ injective? Why or why not?

To employ Theorem KILT, check the kernel of $T$

$$T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \underset{\sim}{0} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\begin{bmatrix}2a-b\\a+3b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ leads to homogeneous system w/ coefficient matrix $\begin{bmatrix}2&-1\\1&3\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1&0\\0&1\end{bmatrix}$

nonsingular!

only solution $a=b=0$

So $\ker(T) = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\} = \{\underset{\sim}{0}\}$ is trivial,

so $T$ is injective.

2. Consider the linear transformation $T\colon S_{22} \to P_2$ defined below, where $S_{22}$ is the vector space of $2\times 2$ symmetric matrices, and $P_2$ is the vector space of polynomials of degree at most 2. (35 points)

$$T\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (3a+b+5c) + (2a+b+4c)\,x + (-3a+4b+5c)\,x^2$$

(a) Compute the kernel of $T$, $\mathcal{K}(T)$.

$T\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = \underset{\sim}{0} = 0 + 0x + 0x^2$ system w/ coeff matrix $\begin{bmatrix} 3 & 1 & 5 \\ 2 & 1 & 4 \\ -3 & 4 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$c$ is free, $a = -c$
$b = -2c$

$$\ker(T) = \left\{ \begin{bmatrix} -c & -2c \\ -2c & c \end{bmatrix} \,\middle|\, c \in \mathbb{C} \right\}$$
$$= \left\{ c\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} \,\middle|\, c \in \mathbb{C} \right\} = \left\langle \left\{ \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} \right\} \right\rangle$$

(b) Compute the range of $T$, $\mathcal{R}(T)$. A basis of the domain is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

By theorem RSLT,

$$R(T) = \left\langle \left\{ T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \right\} \right\rangle = \left\langle \left\{ 3+2x-3x^2,\ 1+x+4x^2,\ 5+4x+5x^2 \right\} \right\rangle$$

This spanning set is linearly dependant since
$(3+2x-3x^2) + 2(1+x+4x^2) = 5+4x+5x^2$.

A basis is $\{3+2x-3x^2,\ 1+x+4x^2\}$

(c) The rank and nullity of $T$ obey a basic relationship. Say what this relationship is, and verify it for $T$.

$$\underbrace{r(T)}_{2 \text{ by part (b)}} + \underbrace{n(T)}_{1 \text{ by part (a)}} = \dim(\text{domain}) = \dim(S_{22}) = 3$$

(d) Compute the preimage of $2 + x - 7x^2$, $T^{-1}\left(2 + x - 7x^2\right)$.

$T\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = 2 + x - 7x^2$ results in system $\left[\begin{array}{ccc|c} 3 & 1 & 5 & 2 \\ 2 & 1 & 4 & 1 \\ -3 & 4 & 5 & -7 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

One element of preimage, with $c=0$
has $a = 1$, $b = -1$

i.e. $T\left(\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\right) = 2 + x - 7x^2$

By Theorem KPI

$$T^{-1}(2+x-7x^2) = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} + K(T)$$
$$= \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} + \left\langle \left\{ \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} \right\} \right\rangle$$

3. For the linear transformation $R\colon M_{22} \to \mathbb{C}^3$ find a specific element of the codomain with an empty pre-image, demonstrating that $R$ is not surjective. ($M_{22}$ is the vector space of $2 \times 2$ matrices.) (15 points)

$$R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2c+2d \\ a-b+3c+d \\ b-c+d \end{bmatrix}$$

We desire $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that $R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ has no solution

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & x \\ 1 & -1 & 3 & 1 & y \\ 0 & 1 & -1 & 1 & z \end{array}\right]$$ should have no solution.

We want $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ not in the column space of the coefficient matrix

Column space of ↑ ; RREF transpose

$$\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\begin{bmatrix} 1 \\ 1 \\ 900 \end{bmatrix}$ is not in the column space,

$$R^{-1}\left(\begin{bmatrix} 1 \\ 1 \\ 900 \end{bmatrix}\right) = \emptyset$$

You could guess almost any vector here & set a system w/ no solution

4. Suppose that $S\colon U \to V$ and $T\colon V \to W$ are linear transformations and each is ~~injective~~. Prove that their composition, $T \circ S$ is ~~invertible~~ injective. (15 points)

Suppose $T(S(u_1)) = T(S(u_2))$

$\Rightarrow S(u_1) = S(u_2)$ since $T$ injective

$\Rightarrow u_1 = u_2$ since $S$ injective.

which says $T \circ S$ is injective.

Suppose $x \in \ker(T \circ S)$

$\Rightarrow T(S(x)) = 0$

$\Rightarrow S(x) = 0$ since $\ker T = \{0\}$

$\Rightarrow x = 0$ since $\ker S = \{0\}$

So $\ker(T \circ S) = \{0\}$

& thus $T \circ S$ is injective by Theorem KILT