The Discrete Fourier Transform:
From Hilbert Spaces to the FFT

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Hilbert Spaces and Hand-waving

Hilbert spaces generalize Euclidean spaces.
What "defines" a Euclidean space:

- Vector space with the dot product
- Calculus

We can generalize the dot product as the inner product

Inner Product

Define the inner product to be a bilinear functional acting on two elements of a vector space $\langle \vec{x}, \vec{y} \rangle$ which is:

- Conjugate symmetric: $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$
- Linear in its first argument: $\langle a\vec{x_1} + b\vec{x_2}, \vec{y} \rangle = a\langle \vec{x_1}, \vec{y} \rangle + b\langle \vec{x_2}, \vec{y} \rangle$
- Positive definite: $\langle \vec{x}, \vec{x} \rangle > 0$ for all $\vec{x} \neq \vec{0}$ and $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$

Hilbert Space and Hand-waving: Inner Product Spaces

A vector space V equipped with an inner product is an Inner Product space.

Norm

We define the norm of a vector in an Inner Product space to be $||\vec{v}||^2 = \langle \vec{v}, \vec{v} \rangle$.

We have some familiar looking results:

Theorem: Parallelogram Law

For $\vec{v_1}, \vec{v_2} \in V$, $||\vec{x}+\vec{y}||^2 + ||\vec{x}-\vec{y}||^2 = 2||\vec{x}||^2 + 2||\vec{y}||^2$

Theorem: Pythagoras

For $\vec{x}, \vec{y} \in V$ such that $\langle \vec{x}, \vec{y} \rangle = 0$, $||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$

Hilbert Spaces and Hand-waving: Metric Spaces

A set M with metric μ is a Metric space if for all $m_1.m_2, m_3 \in M$:

- Identity of Indiscernibles: $\mu(s_1, s_2) = 0$ if and only if $s_1 = s_2$
- Symmetric: $\mu(s_1, s_2) = \mu(s_2, s_1)$
- Triangle Inequality: $\mu(s_1, s_3) \leq \mu(s_1, s_2) + \mu(s_2, s_3)$

$\mu(s_1, s_2)$ is the "distance" between s_1 and s_2.

Completeness

A Cauchy sequence in Metric space M is a sequence $\{m_i\}$ for $i \geq 1$ such that for every $\epsilon > 0$ there exists an N such that $\mu(m_l, m_k) < \epsilon$ for $l, k > N$.
A Metric space M is complete if every Cauchy sequence in M converges to a point in M

Hilbert Spaces and Hand-waving

Theorem

Given an Inner Product space and $\vec{v_1}, \vec{v_2} \in V$, the identity of indiscernibles, the triangle inequality, and symmetry hold for μ defined: $\mu(\vec{v_1}, \vec{v_2}) = ||\vec{v_2} - \vec{v_1}||$.

An Inner Product space which is also a complete Metric space is called a Hilbert space.

We now have:

- Vector space with inner product
- Calculus

Hilbert Spaces and Hand-waving: Central Result

Let $\{\vec{y_i}\}$ be an orthonormal set in Hilbert space $\mathcal{H}$ and $\vec{v} \in \mathcal{H}$ such that $\vec{v} = \sum_k a_k \vec{y_k}$. Then, $a_k = \langle \vec{v}, \vec{y_k} \rangle$

Proof

$$\langle \vec{v}, \vec{y_j} \rangle = \langle \sum_k a_k \vec{y_k}, \vec{y_j} \rangle = \sum_k a_k \langle \vec{y_k}, \vec{y_j} \rangle = a_k$$

Hilbert Spaces and Hand-waving

Example: $\mathbb{R}^2$

$B = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}$ is an orthonormal basis of $\mathbb{R}^2$.

$$\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \left\langle \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \left\langle \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{12}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \frac{4}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 6-2 \\ 6+2 \end{bmatrix}$$

Hilbert Spaces and Hand-waving: $L^2([0,T])$

The set of all complex valued functions with real input integrable on the interval $[0, T]$ such that $\int_0^T |f(x)|^2 dx < \infty$.

Inner Product

$$\langle f(x), g(x) \rangle = \int_0^T f(x)\overline{g(x)}dx$$

Let's examine $\{\frac{1}{\sqrt{T}}e^{i\frac{2\pi}{T}kx} | k \in \mathbb{Z}\}$.
First,

$$\int_0^T |\frac{1}{\sqrt{T}}e^{i\frac{2\pi}{T}kx}|^2 dx = \int_0^T \frac{1}{\sqrt{T}}^2 dx$$
$$= 1 < \infty$$

Next,

$$\langle \frac{1}{\sqrt{T}} e^{i\frac{2\pi}{T}nx}, \frac{1}{\sqrt{T}} e^{i\frac{2\pi}{T}mx} \rangle = \frac{1}{T} \int 0^T e^{i\frac{2\pi}{T}nx} e^{i\frac{2\pi}{T}mx} dx$$

$$= \frac{1}{T} \int 0^T e^{i\frac{2\pi}{T}(n-m)x} dx$$

$$= \begin{cases} \frac{1}{T} \int_0^T 1 dx \text{ for } n = m \\ \frac{1}{i2\pi(n-m)} (e^{2\pi(n-m)} - e^0) \text{ for } n \neq m \end{cases}$$

$$= \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

So, $\{\frac{1}{\sqrt{T}} e^{i\frac{2\pi}{T}kx} | k \in \mathbb{Z}\}$ is orthonormal.

$L^2([0, T])$ and Complex Fourier Series

We can rewrite any $f \in L^2([0, T])$ using our central result:

$$f(x) = \sum_{k \in \mathbb{Z}} a_k e^{i \frac{2\pi}{T} kx}$$

Where:

$$a_k = \langle f, e^{i \frac{2\pi}{T} kx} \rangle = \int_0^T f(x) e^{-i \frac{2\pi}{T} kx} dx$$

This is the complex Fourier series.

Complex Fourier Series: Example

Square wave period 1: $f(x) = \begin{cases} 1 \text{ for } 0 \leq x < \frac{1}{2} \\ -1 \text{ for } \frac{1}{2} \leq x < 1 \end{cases}$

First we find the a_k:

$$\begin{aligned} a_k &= \int_0^1 f(x) e^{-i2\pi kx} dx \\ &= \int_0^{\frac{1}{2}} e^{-i2\pi kx} dx - \int_{\frac{1}{2}}^1 e^{-i2\pi kx} dx \\ &= \frac{-1}{i2\pi k}(e^{-i\pi k} - 1) - \frac{-1}{i2\pi k}(1 - e^{-i\pi k}) \\ &= \frac{-1}{in\pi}(e^{-ik\pi} - 1) \\ &= \begin{cases} 0 \text{ if } k \text{ is even} \\ -\frac{2i}{n\pi} \text{ if } k \text{ is odd} \end{cases} \end{aligned}$$

Fourier Series: Applications

Wide variety of applications:

- Solving PDEs
- Probability theory and statistics
- NMR, IR, etc. spectroscopy
- X-ray crystallography
- MRI
- Image and signal processes
- Audio engineering

Fourier Transform

Generalization to aperiodic functions

Need to consider: $L^2(\mathbb{R})$

Definition: Fourier Transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{i2\pi\omega x}dx$$

And

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i2\pi\omega x}d\omega$$

Orthonormal Basis: $\{e^{i2\pi\omega x} | \omega \in \mathbb{R}\}$

Discrete Fourier Transform

Digital signal and discretized data are common.

Consider a function f sampled uniformly N times over an interval $[0, T]$.
That's at: $0, \frac{T}{N}, 2\frac{T}{N}, \ldots, (N-1)\frac{T}{N}$

"Package" in a vector:

$$f \rightarrow \vec{v} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

Discrete Fourier Transform

Sample basis vectors at the same points.

Roots of Unity

The primitive Nth root of unity is $\omega_N = e^{-i\frac{2\pi}{N}}$

$$e^{i\frac{2\pi}{T}kx} \rightarrow \vec{e}_k = \begin{bmatrix} \overline{\omega_N}^{0\cdot k} \\ \overline{\omega_N}^{1\cdot k} \\ \overline{\omega_N}^{2\cdot k} \\ \vdots \\ \overline{\omega_N}^{(N-1)\cdot k} \end{bmatrix}$$

Note: We only need $\vec{e_0}, \vec{e_1}, \ldots \vec{e}_{N-1}$

Discrete Fourier Transform

We'll call $\hat{\vec{v}}$ the transform of $\vec{v}$.

- $[\hat{\vec{v}}]_k = a_k$

Central result

Let $\{\vec{y_i}\}$ be an orthonormal set in Hilbert space $\mathcal{H}$ and $\vec{v} \in \mathcal{H}$ such that $\vec{v} = \sum_k a_k \vec{y_k}$. Then, $a_k = \langle \vec{v}, \vec{y_k} \rangle$

'Do' the Fourier series on our sampled function:

$$[\hat{\vec{v}}]_k = \langle \vec{v}, \vec{e_k} \rangle = \overline{\vec{e_k}}^* \vec{v}$$

Then,

$$\hat{\vec{v}} = [\vec{e_0} | \vec{e_1} | \dots | \vec{e}_{N-1}]^* \vec{v}$$

The Discrete Fourier Transform

So we define the DFT to be the linear transformation $T : \mathbb{C}^N \to \mathbb{C}^N$ defined by the matrix vector product:

$$T(\vec{v}) = \begin{bmatrix} \omega_N^{0\cdot 0} & \omega_N^{0\cdot 1} & \dots & \omega_N^{0\cdot(N-1)} \\ \omega_N^{1\cdot 0} & \omega_N^{1\cdot 1} & \dots & \omega_N^{1\cdot(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_N^{N\cdot 0} & \omega_N^{N\cdot 1} & \dots & \omega_N^{N\cdot(N-1)} \end{bmatrix} \vec{v}$$

Call this matrix $\mathcal{F}$.

Theorem

The matrix $\mathcal{U} = \frac{1}{\sqrt{N}}\mathcal{F}$ is unitary.

The Discrete Fourier Transform

The DFT is primarily used to go from the ‘time’ to the ‘frequency’ domain

- Spectral analysis
- Spectroscopy
- Filtering
- MRI
  - Spatial information
  - Artifacts
- Audio recording and engineering

Can also be used in data compression

- JPEG
- mp3

Cooley-Tukey Algorithm

Determining F and calculating the matrix vector product is $\mathcal{O}(n^2)$.
We can exploit some symmetries to make this more efficient.

Danielson-Lanczos Lemma

The DFT for $N = 2^m$ for some $m \in \mathbb{N}$, $[\hat{\vec{v}}]_i = \sum_{k=0}^{N-1} [\vec{v}]_k [\mathcal{F}_N]_{ik}$, may be rewritten

$$[\hat{\vec{v}}]_i = [\mathcal{F}_{\frac{N}{2}} \vec{v}_{\text{even}}]_i + [D_{\frac{N}{2}} \mathcal{F}_{\frac{N}{2}} \vec{v}_{\text{odd}}]_i \text{ for } 0 \leq i \leq \frac{N}{2} - 1$$

$$[\hat{\vec{v}}]_i = [\mathcal{F}_{\frac{N}{2}} \vec{v}_{\text{even}}]_i - [D_{\frac{N}{2}} \mathcal{F}_{\frac{N}{2}} \vec{v}_{\text{odd}}]_i \text{ for } \frac{N}{2} < i < N - 1$$

Cooley-Tukey Algorithm

Theorem

Given an $N-$point DTF, $\mathcal{F}_N$, where $N = 2^k$ where $k \in \mathbb{N}$. Then,

$$\mathcal{F}_N = \begin{bmatrix} I_{\frac{N}{2}} & D_{\frac{N}{2}} \\ I_{\frac{N}{2}} & -D_{\frac{N}{2}} \end{bmatrix} \begin{bmatrix} \mathcal{F}_{\frac{N}{2}} & \\ & \mathcal{F}_{\frac{N}{2}} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}.$$

Then the FFT may be calculated

- Calculate the ω_N ($\mathcal{O}(N)$)
- Recursively apply this decomposition $\log_2 N$ times

This recursion gives $\log_2 N$ operations for each 'slot' and there are N slots so we have $\mathcal{O}(N \log N)$

Conclusions

1. Euclidean spaces can be generalized to Hilbert spaces
2. Square-integrable functions are vectors in the Hilbert space $L^2(\mathbb{R})$ and can be expressed as a linear combination of basis vectors
3. The Fourier series and Fourier Transform are vector decomposition with the special basis $\{e^{i2\pi\omega x}\}$
4. The DFT can 'do' the Fourier Transform on discrete data and can be represented as a matrix vector product
5. The DFT can be more efficiently calculated using the Cooley-Tukey Algorithm

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