Math 181 Friday, April 2 Section 10.7/10.8

Mon- 10.8

Tue- 8.1

No 8.1- preview

Taylor Polynomials

$f(x)$ - function, $g(x)$ - approximating polynomial

$$g(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots$$

what should $b_0, b_1, b_2, b_3, \ldots$ be?

Strategy: match first, second, third, ... derivatives of $f(x)$ & $g(x)$ at $\underline{\underline{x=a}}$.

$$f(a) = g(a) = b_0 + b_1(a-a) + b_2(a-a)^2 + \cdots + \quad = b_0$$

$$g'(x) = b_1 + 2b_2(x-a) + 3b_3(x-a)^2 + \cdots$$

$$f'(a) = g'(a) = b_1 + 2b_2(a-a) + 3b_3(a-a)^2 + \cdots \quad = b_1$$

$$g''(x) = 2b_2 + 3\cdot 2 b_3(x-a) + 4\cdot 3 b_4(x-a)^2 + \cdots$$

$$f''(a) = g''(a) = 2b_2 + 3\cdot 2 b_3(a-a) + 4\cdot 3 b_4(a-a)^2 + \cdots = 2b_2 \rightarrow b_2 = \frac{1}{2}f''(a)$$

$$g'''(x) = 3\cdot 2\, b_3 + 4\cdot 3\cdot 2\, b_4 (x-a) + 5\cdot 4\cdot 3\, b_5 (x-a)^2 + \cdots$$

$$f'''(a) = g'''(a) = 3\cdot 2\, b_3 + 4\cdot 3\cdot 2\, b_4 (a-a) + 5\cdot 4\cdot 3\, b_5 (a-a)^2 + \cdots = 3\cdot 2\cdot b_3$$

$$\rightarrow b_3 = \frac{1}{3\cdot 2} f'''(a)$$

$$b_4 = \frac{1}{4\cdot 3\cdot 2} f^{(4)}(a) \qquad b_n = \frac{1}{n!} f^{(n)}(a)$$

<u>Defn</u> n^{th} <u>Taylor</u> <u>polynomial</u> for $f(x)$ at $x=a$.

$$T_n(x) = \boxed{\underbrace{f(a) + f'(a)(x-a)}_{\text{tangent line}} + \frac{f''(a)}{2!}(x-a)^2} + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

approximating quadratic

McLaurin polynomial, case <u>$a=0$</u>

Ex $f(x) = \cos(x) \qquad a = 0$

$f(x) = \cos(x) \qquad f(0) = 1$

$f'(x) = -\sin(x) \qquad f'(0) = 0$

$f''(x) = -\cos(x) \qquad f''(0) = -1$

$f^{(3)}(x) = \sin(x) \qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x) \qquad f^{4}(0) = 1$

$f^{(5)}(x) = -\sin(x) \qquad f^{(5)}(0) = 0$

$f^{(6)}(x) = -\cos(x) \qquad f^{(6)}(0) = -1$

$f^{(7)}(x) = \sin(x) \qquad f^{(7)}(0) = 0$

$(x-0)$

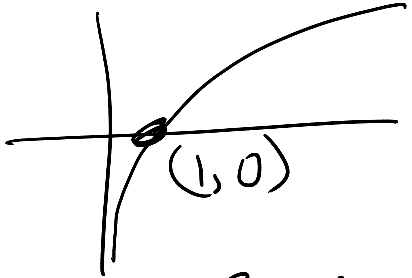
$$T_n(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$= \sum_{k=0}^{n/2} \frac{(-1)^k}{(2k)!} x^{2k}$$

∞

radius of convergence

Ex $f(x) = \ln(x)$, $a = 1$



$f(x) = \ln(x)$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\qquad f'(1) = 1$

$f''(x) = (-1)x^{-2} = \frac{-1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = (-1)(-2)x^{-3} = \frac{2}{x^3}$ $\qquad f^{(3)}(1) = 2!$

$f^{(4)}(x) = (-1)(-2)(-3)x^{-4} = \frac{-3!}{x^4}$ $\qquad f^{(4)}(1) = -3!$

$f^{(5)}(x) = \qquad = \frac{4!}{x^5}$ $\qquad f^{(5)}(1) = 4!$

$$T_n(x) = 0 + 1(x-1) + \frac{(-1)}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \frac{4!}{5!}(x-1)^5 + \cdots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 + \cdots$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}(x-1)^k$$

As a power series, limited radius of convergence.

Ex $f(x) = e^x$, $a = 0$, Taylor Polynomials

$\sin(x^2)$

$\sin$ $2x + x^2 \cos$

$f(x) = e^x$ $f(0) = e^0 = 1$

$f'(x) = e^x$ $f'(0) = e^0 = 1$

$f''(x) = e^x$ $f''(0) = e^0 = 1$

$f^{(3)}(x) = e^x$ $f^{(3)}(0) = e^0 = 1$

$f^{(4)}(x) = e^x$ $f^{(4)}(0) = e^0 = 1$

$f^{(5)}(x) = e^x$ $f^{(5)}(0) = e^0 = 1$

$\vdots$

$$T_n(x) = 1 + 1(x-0) + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \cdots + \frac{1}{n!}(x-0)^n$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

$$= \sum_{k=0}^{n} \frac{1}{k!}x^k$$

[Let $n \to \infty$, radius of convergence]