Math 390 Friday, February 12 SCLA Direct Sums, Orthogonal Complements

FCLA: Section S

Mon - Review MR.TFO

Tue - Exam

BREAK

Mon - Jordan Canonical Form

Defn U & W subspaces of V

$$U+W = \{ \underset{\sim}{u}+\underset{\sim}{w} \mid \underset{\sim}{u} \in U, \underset{\sim}{w} \in W \}$$

theorem SSIS $U+W$ subspace

theorem SID

$$\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

used in theorem UTEC

Defn Direct Sum

V - vector space

U, W subspaces

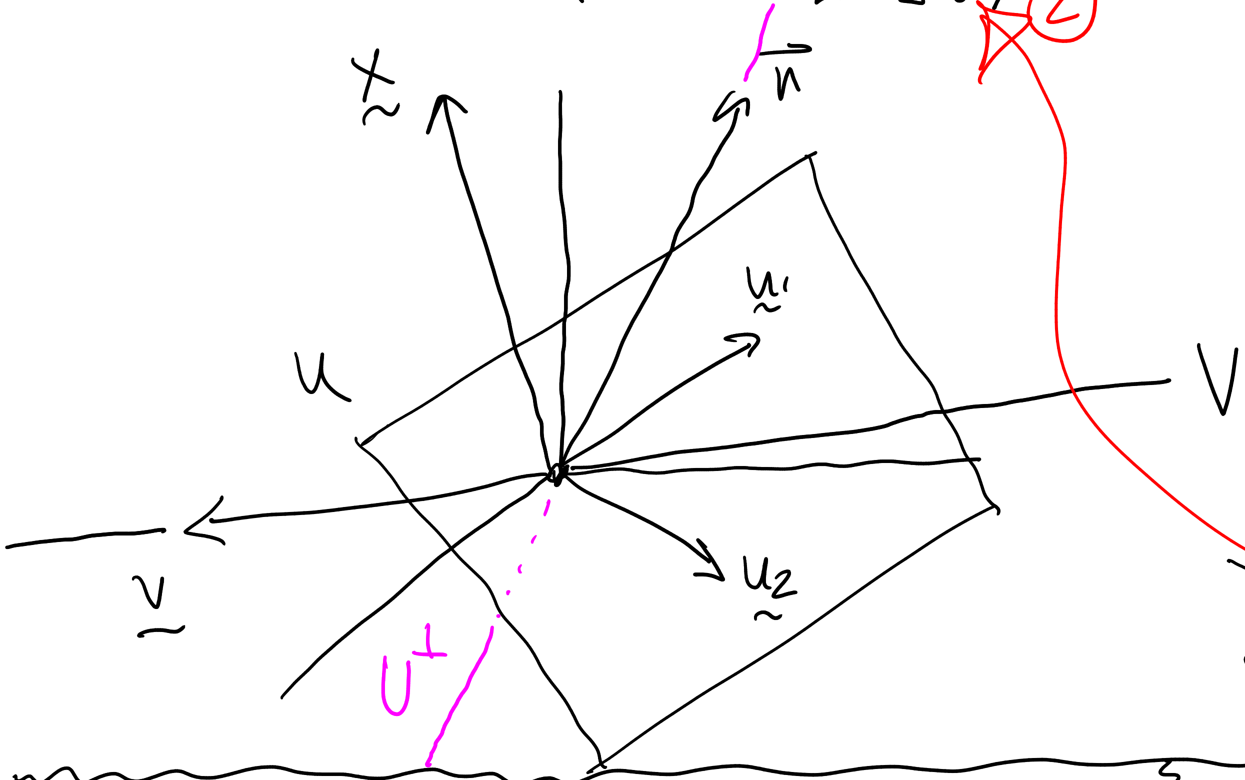
we say V is a direct sum of U & W, written $V = U \oplus W$

if ① For every $\underset{\sim}{v} \in V$ there is $\underset{\sim}{u} \in U$, $\underset{\sim}{w} \in W$ so that $\underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{w}$.

② If $\underset{\sim}{v} = \underset{\sim}{u_1} + \underset{\sim}{w_1}$, $\underset{\sim}{v} = \underset{\sim}{u_2} + \underset{\sim}{w_2}$ then $\underset{\sim}{u_1} = \underset{\sim}{u_2}$ & $\underset{\sim}{w_1} = \underset{\sim}{w_2}$

"① is unique"

Ex $\mathbb{R}^3$ $\quad U = \left\langle \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \right\} \right\rangle$ ② $\qquad V = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \right\rangle \quad v_1 \notin U$



Fact $\{u_1, u_2, v_1\}$ basis of $\mathbb{R}^3$

$x \in \mathbb{R}^3 \qquad x = \underbrace{(a_1 u_1 + a_2 u_2)}_{\in U} + \underbrace{a_3 v_1}_{\in V}$

VRRB $\Rightarrow$ a_1, a_2, a_3 unique

$\Rightarrow$ are unique

So V is the/a complement of U

Line normal to plane (280) $\quad U^\perp = \left\langle \left\{ \begin{bmatrix} -10 \\ 5 \\ 5 \end{bmatrix} \right\} \right\rangle$

normal vector

$$u_1 \times u_2 = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ -1 & 1 & -3 \end{vmatrix} = \begin{bmatrix} -10 \\ 5 \\ 5 \end{bmatrix} = n$$

$U = C\left(\begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 1 & -3 \end{bmatrix} \right), \quad U^\perp = N\left(\begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & -3 \end{bmatrix} \right)$ ①

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$N(\ ) = \left\langle \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

Theorem 1.2.3 $\{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_n}\}$ basis of V.

$$V = \langle \{\underset{\sim}{v_1}, \dots, \underset{\sim}{v_t}\} \rangle \oplus \langle \{\underset{\sim}{v_{t+1}}, \dots, \underset{\sim}{v_n}\} \rangle$$

Theorem 1.2.4 V vector space, U subspace $\Rightarrow$ there exists W so that $V = U \oplus W$

Proof Extend a basis of U, to basis (spanning set) for W

Defn W "complement of V"

Theorem 1.2.5 Replace condition 2 by $\underset{\sim}{u} + \underset{\sim}{w} = \underset{\sim}{0} \Rightarrow \underset{\sim}{u} = \underset{\sim}{w} = \underset{\sim}{0}$

Theorem 1.2.6 Replace 2 by $U \cap W = \{\underset{\sim}{0}\}$

Theorem SID $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

1 EZ

2 1.2.6

$\dim(U \oplus W) = \dim(U) + \dim(W)$

0

Orthogonal complements

Given U, define orthogonal complement, $U^\perp$ ("U-perp")

$$U^\perp = \{ \underset{\sim}{v} \mid \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle = 0 \}$$ (inner products in FCLA)

$V = U \oplus U^\perp$ theorem 1.3.7

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theorem 1.3.6 U subspace of V, A matrix col columns that span U

$$\Rightarrow N(A^*) = U^\perp$$

$$\underset{\sim}{v} \in U^\perp \iff \underset{\sim}{v}^* A = \underset{\sim}{0}^* \overset{*}{\iff} A^* \underset{\sim}{v} = \underset{\sim}{0} \iff \underset{\sim}{v} \in N(A^*)$$

Corollary 1.3.8
A $m \times n$ matrix

$$\underbrace{\mathbb{C}^m = \mathcal{C}(A) \oplus N(A^*)}$$