Math 390 Thursday, February 25 Jordan Canonical Form

$$J_3(5) = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \qquad J_3(5) - 5I_3 = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

nilpotent

Thu JCF
Fri JCF
Movies

Defn Jordan Canonical Form

A matrix that

1. Block diagonal
2. Each block is Jordan block
3. If λ & ρ eigenvalues w/ $\lambda > \rho$ then $J(\lambda)$ is "higher" than $J(\rho)$
4. For each eigenvalue, larger blocks are higher.

Ex

$$\begin{array}{ccc} \times 1 & \times 4 & \times 6 \\ \downarrow & \downarrow & \downarrow \end{array}$$

$$\left[\begin{array}{ccc|cc|cc} -1 & 1 & & & & & \\ & -1 & 1 & & & & \\ & & -1 & & & & \\ \hline & & & -1 & 1 & & \\ & & & & -1 & & \\ \hline & & & & & -5 & 1 \\ & & & & & & -5 \end{array}\right]$$

$\alpha(-1) = 5$ $\quad$ $\alpha(-5) = 2$

$\gamma(-1) = 2$ $\quad$ $\gamma(-5) = 1$

$c(-1) = 3$ $\quad$ $c(-5) = 2$

Theorem 3.3.5 Every linear transformation $T: V \to V$ has a basis that yields a matrix representation in JCF

Theorem 3.3.1 main argument

PLUS

- λ eigenvalue of $T \iff J_k(\lambda)$ is a block
- Largest block for eigenvalue λ has size $\iota(\lambda)$ ← index
- Number of blocks for an eigenvalue is $\gamma(\lambda)$ ← geom. mult.
- Total of sizes for all blocks is $\alpha(\lambda)$ ← alg. mult.

Ex

Jordan Chains $T: V \to V$, λ eigenvalue of index p, $K_i = K((T-\lambda I)^i)$

$\underset{\sim}{x}_p \in K_p \setminus K_{p-1}$ $\quad \underset{\sim}{x}_{p-1} = (T-\lambda I)\underset{\sim}{x}_p$, $\underset{\sim}{x}_{p-2} = (T-\lambda I)\underset{\sim}{x}_{p-1}$

.... $\underset{\sim}{x}_{k-1} = (T-\lambda I)\underset{\sim}{x}_k$

Facts

1. $\underset{\sim}{x}_k \in K_k$

$$(T-\lambda I)^k \underset{\sim}{x}_k = (T-\lambda I)^k (T-\lambda I)\underset{\sim}{x}_{k+1}$$

$$= (T-\lambda I)^{k+1} \underset{\sim}{x}_{k+1}$$

$$= (T-\lambda I)^{k+1} (T-\lambda I)\underset{\sim}{x}_{k+2}$$

$$= (T-\lambda I)^{k+2} \underset{\sim}{x}_{k+2} = \cdots = (T-\lambda I)^p \underset{\sim}{x}_p = \underset{\sim}{0}$$

$\underset{\sim}{x}_p \in K_p$

② $\underset{\sim}{x}_1$ is a traditional eigenvector

$$\underset{\sim}{x}_1 \in K_1 = K((T-\lambda I)^1) = K(T-\lambda I) = \mathcal{E}_T(\lambda)$$

$\underset{\sim}{x}_0 = \underset{\sim}{0}$

③ $$\underset{\sim}{x}_{k-1} = (T-\lambda I)\underset{\sim}{x}_k$$

$$= T\underset{\sim}{x}_k - \lambda \underset{\sim}{x}_k$$

Rearrange

$$T\underset{\sim}{x}_k = \underline{1\,\underset{\sim}{x}_{k-1}} + \underline{\lambda \underset{\sim}{x}_k}$$

$$(T\underset{\sim}{x}_1 = \lambda \underset{\sim}{x}_1)$$

pesky 1 $\rightarrow$ $1\,\underset{\sim}{x}_{k-1}$

$\uparrow$ almost diagonal