Math 390 Thursday, March 18 SVD

Theorem 2.3.1 (SCLA) Exam #2 April 6

$A_{m\times n}$

r = rank

	δ_1	δ_2		δ_r	0		$0 \quad 0$	eigenvalues
A^*A	$\underset{\sim}{x}_1$	$\underset{\sim}{x}_2$	$\cdots$	$\underset{\sim}{x}_r$	$\underset{\sim}{x}_{r+1}$	$\cdots$	$\underset{\sim}{x}_n$	orthonormal basis of eigenvectors
AA^*	$\underset{\sim}{y}_1$	$\underset{\sim}{y}_2$	$\cdots$	$\underset{\sim}{y}_r$	$\underset{\sim}{y}_{r+1}$	$\cdots$	$\underset{\sim}{y}_m$	orthonormal basis of eigenvectors

$A\underset{\sim}{x}_i = \sqrt{\delta_i}\,\underset{\sim}{y}_i \quad , 1 \le i \le r$ $\qquad A\underset{\sim}{x}_i = \underset{\sim}{0} \quad r+1 \le i \le n$

singular values of A

eigenvector of AA^*

Proof

A^*A & AA^* normal matrices $\Rightarrow$ diagonalizable via an orthonormal basis of eigenvectors (Theorem DMFE)

Implies algebraic multiplicities equal geometric multiplicities

Grab $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n\}$ as orthonormal basis of eigenvectors for A^*A.
Order as above.

λ eigenvalue of A^*A, and hence an eigenvalue of AA^*

$\mathcal{E}_{A^*A}(\lambda) = \langle \{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_s\} \rangle$ orthogonal basis of eigenspace (s = geom mult λ for A^*A)

eigenvectors of A^*A

$\langle A\underset{\sim}{x}_i, A\underset{\sim}{x}_j \rangle = \langle \underset{\sim}{x}_i, A^*A\underset{\sim}{x}_j \rangle = \langle \underset{\sim}{x}_i, \lambda \underset{\sim}{x}_j \rangle = \lambda \langle \underset{\sim}{x}_i, \underset{\sim}{x}_j \rangle = \lambda 0 = 0$

So $\{A\underset{\sim}{x}_1, A\underset{\sim}{x}_2, \ldots, A\underset{\sim}{x}_s\}$ orthogonal (hence linearly independent) in $\mathcal{E}_{AA^*}(\lambda)$

Thus $\underbrace{\dim(\mathcal{E}_{A^*A}(\lambda))}_{s} \leq \dim(\mathcal{E}_{AA^*}(\lambda))$: $\gamma_{A^*A}(\lambda) \leq \gamma_{AA^*}(\lambda)$

So $\alpha_{A^*A}(\lambda) = \gamma_{A^*A}(\lambda) \leq \gamma_{AA^*}(\lambda) = \alpha_{AA^*}(\lambda)$

Same formula for A^* (in place of A)

$\alpha_{(A^*)^*A^*}(\lambda) \leq \alpha_{A^*(A^*)^*}(\lambda) \longrightarrow \alpha_{AA^*}(\lambda) \leq \alpha_{A^*A}(\lambda)$

$\Rightarrow \alpha_{AA^*}(\lambda) = \alpha_{A^*A}(\lambda)$ (geom mult's too)

rank + nullity = # columns $\rightarrow$ (# non zero eigenvalues) + (number of zero eigenvalues)
= # cols
number = algebraic multiplicity

$$\operatorname{rank}(A^*A) = \sum_{\lambda \neq 0} \alpha_{A^*A}(\lambda)$$

equal

$$\operatorname{rank}(AA^*) = \sum_{\lambda \neq 0} \alpha_{AA^*}(\lambda)$$

equal algebraic multiplicities for nonzero eigenvalues
$\Rightarrow$ same number of distinct nonzero eigenvalues $(p=q)$

Define a set of eigenvectors for AA^*

$$\underset{\sim}{y}_i = \frac{1}{\sqrt{\sigma_i}} A\underset{\sim}{x}_i \qquad 1 \leq i \leq r$$

nonzero

already an eigenvector of AA^*

choose $\underset{\sim}{y}_i$, $r+1 \leq i \leq m$
orthonormal basis of $N(AA^*)$
Gram-Schmidt

$$\left(\text{so } \quad A\underset{\sim}{x}_i = \sqrt{\sigma_i}\, \underset{\sim}{y}_i \right)$$

Fact $\underset{\sim}{y}_i$ an orthonormal set.

orthogonality is EZ $\langle y_i, y_j \rangle = \langle \frac{1}{\sqrt{\delta_i}} A x_i, \frac{1}{\sqrt{\delta_j}} A x_j \rangle$

$$= \frac{1}{\sqrt{\delta_i}} \frac{1}{\sqrt{\delta_j}} \langle x_i, A^* A x_j \rangle$$

$$= \frac{1}{\sqrt{\delta_i}} \frac{1}{\sqrt{\delta_j}} \langle x_i, \delta_j x_j \rangle$$

$$= \frac{1}{\sqrt{\delta_i}} \frac{1}{\sqrt{\delta_j}} \delta_j \quad 0$$

Norm 1?

$$\|y_i\|^2 = \langle y_i, y_i \rangle = \left\langle \frac{1}{\sqrt{\delta_i}} A x_i, \frac{1}{\sqrt{\delta_i}} A x_i \right\rangle$$

$$= \frac{1}{\sqrt{\delta_i}} \frac{1}{\sqrt{\delta_i}} \langle x_i, A^* A x_i \rangle = \frac{1}{\delta_i} \langle x_i, \delta_i x_i \rangle$$

$$= \frac{1}{\delta_i} \delta_i \langle x_i, x_i \rangle = \|x_i\|^2 = 1^2 = 1$$

Punchline $U = [y_1 | \cdots | y_n]$, $V = [x_1 | x_2 | \cdots | x_m] \Rightarrow A = U \begin{bmatrix} \sqrt{\delta_1} & & & 0 \\ & \sqrt{\delta_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\delta_r} \end{bmatrix} V^*$