Math 390  Friday, March 26

Theorem OD  A square
Unitary U, diagonal D so that
\[ U^*AU = D \iff A \text{ is normal} \]

Comment: columns of U are orthonormal basis of eigenvectors of A
- Diagonal elements of D are eigenvalues of A

Proof \((\Rightarrow)\)
\[ A^*A = (UDU^*)(UDU^*) \]
\[ U^*AU = D \]
\[ \Rightarrow A = UDU^* \]

\[ U^*DU^* \]
\[ = UDD^*U^* \]
\[ = UDDU^* \]
\[ = UDU^*UDU^* \]

\[ \text{diagonal matrices commute} \]
\[ = U D U^* (U D U^*)^* \]

\[ \left(\Leftarrow \right) \text{ Know } A \text{ is normal} \]

Then \( OBUTR \Rightarrow \text{ Unitary } U \) so that \( U^* A U = T \) \( \leftarrow \text{ upper-triangular} \)

Claim \( T \) is also normal

\[ T^* T = (U^* A U)^* (U^* A U) \]

\[ = U^* A^* U U^* A U \]

\[ = U^* A^* A U \]

\[ = U^* A A^* U \]

\[ = U^* A U U^* A^* U \]

\[ = (U^* A U) (U^* A U)^* \]

\[ = T^* T \]
Fact \( TT^* - T^*T = 0 \)

\[
0 = [TT^* - T^*T]_{ii} = [T - T^*]_{ii} - [T^*T]_{ii}
\]

\[
= \sum_{k=1}^{n} [T]_{ik} [T^*]_{ki} - \sum_{k=1}^{n} [T^*]_{ik} [T]_{ki}
\]

\[
= \sum_{k=1}^{n} [T]_{ik} [T]_{ik} - \sum_{k=1}^{n} [T]_{ki} [T]_{ki}
\]

\[
= \sum_{k=1}^{n} |[T]_{ik}|^2 - \sum_{k=1}^{n} |[T]_{ki}|^2
\]

\( T \) upper-triangular positive

\( (a+bi)(a-bi) = a^2 + b^2 \)

\( |a+bi| = \sqrt{a^2 + b^2} \)

Now determine that entries of \( T \), above the diagonal, are all zero, starting with row \( i = 1 \).
Repeat, in order, for each subsequent row:

Second row is zeros to the left of the diagonal.

First row is (get diagonal)

\[
\begin{bmatrix}
T_{21} & 0 & \ldots & 0 \\
T_{23} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
T_{2n} & \ldots & 0 & 0
\end{bmatrix}
\]

For each row:

Sum of positive quantities = 0

\[
0 = \sum_{k=1}^{n} |T_{2k}|
\]

For each row:

\[
|T_{21}| - \sum_{k=2}^{n} |T_{2k}| = 0
\]

\[
\begin{align*}
T_{11} & = -2 \\
T_{12} & = 0 \\
T_{13} & = -3 |T_{12}|k = 2 &= 2 |T_{12}| \\
T_{14} & = 0 \\
T_{15} & = 3 |T_{13}|k = 3 &= 3 |T_{13}| \\
\vdots & \quad \vdots \\
T_{1n} & = \ldots \quad \vdots \\
T_{21} & = 0 \\
T_{22} & = -3 |T_{21}|k = 2 &= 2 |T_{21}| \\
T_{23} & = 0 \\
T_{24} & = -3 |T_{23}|k = 3 &= 3 |T_{23}| \\
\vdots & \quad \vdots \\
T_{2n} & = \ldots \quad \vdots \\
\end{align*}
\]