Math 390 — Monday, April 12 — Least-Squares, Projectors

Tue - Problem Session

Projectors (SCLA 1.6)

Determinants

New problem / Least Squares Application

" R^2 " goodness-of-fit

Solving normal equations $(A\underset{\sim}{x} = \underset{\sim}{b})$

$$A^* A \hat{\underset{\sim}{x}} = A^* \underset{\sim}{b}$$

Suppose A has full rank &

$A = QR$ is QR decomposition

(R invertible)

$$R\hat{\underset{\sim}{x}} = \underbrace{(R^*)^{-1} R^*}\ \underbrace{Q^* Q}\ R\hat{\underset{\sim}{x}}$$

upper-triangular, backsolve for $\hat{\underset{\sim}{x}}$

$$= (R^*)^{-1} (QR)^* (QR) \hat{\underset{\sim}{x}}$$

$$= (R^*)^{-1} A^* A \hat{\underset{\sim}{x}}$$

normal equations

$$= (R^*)^{-1} A^* \underset{\sim}{b}$$

$$= (R^*)^{-1} (QR)^* \underset{\sim}{b} = (R^*)^{-1} R^* Q^* \underset{\sim}{b} = Q^* \underset{\sim}{b}$$

unitary, well behaved numerically

Projectors

$N^4 = 0$

oblique

Defn A square matrix P is a projector if $P^2 = P$. "idempotent"

$$P^3 = P \cdot P^2 = P \cdot P = P^2 = P, \ \ldots, \ P^{832} = P$$

$P\underset{\sim}{x}$ takes $\underset{\sim}{x}$ into $C(P)$, P "fixes" every $\underset{\sim}{x} \in C(P)$

Theorem P projector, $\underset{\sim}{x} \in C(P) \Rightarrow P\underset{\sim}{x} = \underset{\sim}{x}$; $P\underset{\sim}{x} - \underset{\sim}{x} = \underset{\sim}{0}$

Proof $\underset{\sim}{x} \in C(P) \Rightarrow$ there is a $\underset{\sim}{w}$ so that $P\underset{\sim}{w} = \underset{\sim}{x}$

① $P\underset{\sim}{x} - \underset{\sim}{x} = P(P\underset{\sim}{w}) - P\underset{\sim}{w} = P^2\underset{\sim}{w} - P\underset{\sim}{w} = P\underset{\sim}{w} - P\underset{\sim}{w} = \underset{\sim}{0}$

② $P\underset{\sim}{x} = P(P\underset{\sim}{w}) = P^2\underset{\sim}{w} = P\underset{\sim}{w} = \underset{\sim}{x}$

Theorem P projector, $\mathbf{x}$ any vector $\Rightarrow P\mathbf{x} - \mathbf{x} \in N(P)$

interested in $\mathbf{x} \notin C(P)$

$= \mathbf{0}$ if $\mathbf{x} \in C(P)$

$$N(P) = \{P\mathbf{x} - \mathbf{x} \mid \mathbf{x} \in \mathbb{C}^n\}$$

Proof First,

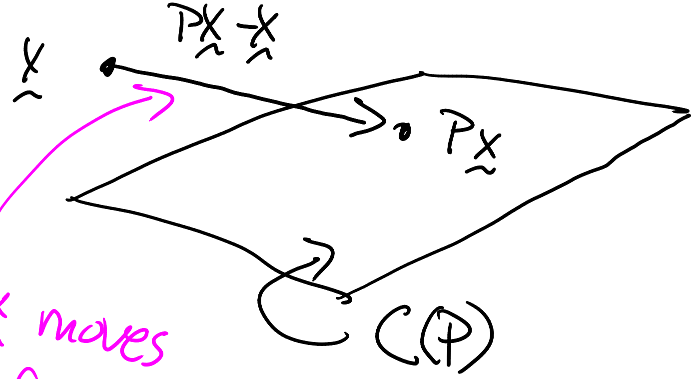
$$P(P\mathbf{x} - \mathbf{x}) = P^2\mathbf{x} - P\mathbf{x} = P\mathbf{x} - P\mathbf{x} = \mathbf{0} \Rightarrow P\mathbf{x} - \mathbf{x} \in N(P)$$

Know $\{P\mathbf{x} - \mathbf{x} \mid \mathbf{x} \in \mathbb{C}^n\} \subseteq N(P)$

Grab $\mathbf{z} \in N(P)$ so

$$\mathbf{z} = \mathbf{0} - (-\mathbf{z}) = P(-\mathbf{z}) - (-\mathbf{z}) \in \{P\mathbf{x} - \mathbf{x} \mid \mathbf{x} \in \mathbb{C}^n\}$$

So we have set equality



direction $\mathbf{x}$ moves in a direction from $N(P)$

Complementary Projectors

Defn P projector, then $I-P$ is the complementary projector.

Theorem $I-P$ is a projector.

Proof
$$
\begin{aligned}
(I-P)^2 &= (I-P)(I-P)\\
&= I^2 - P - P + P^2\\
&= I - P - P + P\\
&= I - P
\end{aligned}
$$

Theorem P projector $C(I-P) = N(P)$ $\quad \left[N(I-P) = C(I-(I-P)) = C(P) \right]$

Proof $N(P) \subseteq C(I-P)$

Grab $\underset{\sim}{x} \in N(P)$ $\quad (I-P)\underset{\sim}{x} = I\underset{\sim}{x} - P\underset{\sim}{x} = \underset{\sim}{x} - \underset{\sim}{0} = \underset{\sim}{x} \Rightarrow \underset{\sim}{x} \in C(I-P)$

$\underset{\sim}{x}$ is in column space of

$C(I-P) \subseteq N(P)$

Grab $\underset{\sim}{x} \in C(I-P)$, so there exists $\underset{\sim}{w}$ so that $(I-P)\underset{\sim}{w} = \underset{\sim}{x}$

Look at

$$P\underset{\sim}{x} = P(I-P)\underset{\sim}{w} = P\underset{\sim}{w} - P^2\underset{\sim}{w} = P\underset{\sim}{w} - P\underset{\sim}{w} = \underset{\sim}{0}$$

So $\underset{\sim}{x} \in N(P)$.

<u>Theorem</u> P projector $\mathbb{C}^n = C(P) \oplus N(P)$

<u>Proof</u> 1 $C(P) \cap N(P) = \{\underset{\sim}{0}\}$ Grab $\underset{\sim}{x} \in C(P)$ & $\underset{\sim}{x} \in N(P)$

$\underset{\sim}{x} \in C(P) \Rightarrow \underset{\sim}{x} \in N(I-P)$

So

$$\underset{\sim}{x} = \underset{\sim}{x} - \underset{\sim}{0} = \underset{\sim}{x} - P\underset{\sim}{x} = (I-P)\underset{\sim}{x} = \underset{\sim}{0}$$

$\underset{\sim}{x} \in N(P)$ $\qquad$ $\underset{\sim}{x} \in N(I-P)$

② show every vector can be written as sum of vectors from $C(P)$, $N(P)$

Grab $\underset{\sim}{w} \in \mathbb{C}^n$,

$$\underset{\sim}{w} = I\underset{\sim}{w} - P\underset{\sim}{w} + P\underset{\sim}{w} = \underbrace{(I-P)\underset{\sim}{w}}_{\text{in } C(I-P) = N(P)} + \underbrace{P\underset{\sim}{w}}_{\text{in } C(P)}$$

Eigenvalues of a projector?