\[ P_A(x) = \det \left( xI - A \right) \quad (290) \]
\[ = \prod_{i=1}^{n} (x - \lambda_i) \quad (390) \]

\[ P_A(0) = \det (0I - A) = \det (-A) = (-1)^n \det (A) \]

\[ P_A(0) = \prod_{i=1}^{n} (0 - \lambda_i) = (-1)^n \prod_{i=1}^{n} \lambda_i \]

This could be the definition of the determinant.

**Defn**: A square matrix, the **trace** of \( A \), is the sum of the diagonal entries, \( \text{tr}(A) = \sum_{i=1}^{n} [A]_{ii} \).
Given \( A \), find \( T \) that is upper-triangular & similar to \( A \). Then \( \text{SU}_T \) or \( \text{UTEC} \), says the diagonal entries are eigenvalues.

\[
\text{tr}(T) = \sum_{i=1}^{n} \lambda_i
\]

\[
\rho_{A}(x) = \prod_{i=1}^{n} (x - \lambda_i) = (x - \lambda_1)(x - \lambda_2) - \cdots (x - \lambda_n)
\]

\[
= x^n + (-\lambda_1 \cdots - \lambda_n)x^{n-1} + \cdots
\]

\[
= x^n - \text{tr}(T)x^{n-1}
\]

Any matrix similar to \( T \) (notably \( A \)) will have the same characteristic polynomial, so the trace is always the negative of the sum of the eigenvalues.

Fact: \( \det(A) \) & \( \text{tr}(A) \) are properties of a linear transformation, since they are the same for every matrix representation.
Determine Definition of a Determinant

**Defn.** A permutation is a 1-1 onto function from a finite set to itself.

**Ex.** \( X = \{1, 2, 3, 4\} \)

\[ f: \begin{align*}
1 &\rightarrow 2 \\
2 &\rightarrow 3 \\
3 &\rightarrow 1
\end{align*} \]

\[ \begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 1 & \uparrow
\end{pmatrix} \]

\[ \text{domain} \quad \text{codomain} \]

\[ \Rightarrow \text{rearrangement} \]

\[ \Rightarrow \text{"natural" order} \]

**Ex.** \( X = \{a, b, c, d\} \)

Permutations:

\[ \begin{align*}
&ba \quad ac \\
&cb \quad ad \\
&ab \quad cd \\
\end{align*} \]

"Fixes" \( b \) \& \( d \)

Identity permutation:

\[ abcd \]

Inverse: \( bdac \) \& composition has inverse \( cadb \) is the identity function.
Define: Suppose $A = (a_{ij})$ is an $n \times n$ matrix. Sn set of all permutations of $1, 2, \ldots, n$. Then

$$\det (A) = \sum_{\sigma \in S_n} \text{sign} (\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$$

with $\text{sign}(\sigma) \in \{+1, -1\}$,

$$\begin{vmatrix}
2 & 1 & 2 \\
3 & -2 & 1 \\
-1 & 2 & 1
\end{vmatrix}
$$

$\sigma = 312$ (1 $\rightarrow$ 3, 2 $\rightarrow$ 1, 3 $\rightarrow$ 2)

$\sigma_{13} a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$

$= 2 \cdot (-1) \cdot (2) = -4$